The inversion formula and holomorphic extension of the minimal representation of the conformal group

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Dedicated to Roger Howe on the occasion of his 60th birthday

Abstract

The minimal representation π of the indefinite orthogonal group $O(m+1,2)$ is realized on the Hilbert space of square integrable functions on $\mathbb{R}^m$ with respect to the measure $|x|^{-1}dx_1\cdots dx_m$. This article gives an explicit integral formula for the holomorphic extension of π to a holomorphic semigroup of $O(m+3,\mathbb{C})$ by means of the Bessel function. Taking its 'boundary value', we also find the integral kernel of the 'inversion operator' corresponding to the inversion element on the Minkowski space $\mathbb{R}^{m,1}$.

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Contents

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1 Introduction

1.1 Semigroup generated by a differential operator D

Consider the differential operator

$$\begin{aligned} D &:= |x|\Big(\frac{\Delta}{4}-1\Big) \\ &= \frac{1}{4}\Big(\sum_{j=1}^{m} x_j^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{m}\frac{\partial^2}{\partial x_j^2}-4\Big) \end{aligned} \tag{1.1.1}$$

on $\mathbb{R}^m$. A distinguishing feature here is that D has the following properties (see Remark 3.4.4):

1) D extends to a self-adjoint operator on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$.

2) D has only discrete spectra $\{-(j+\frac{m-1}{2}) : j = 0, 1, \cdots\}$ in $L^2(\mathbb{R}^m, \frac{dx}{|x|})$.
Therefore, one can define a continuous operator

$$e^{tD} = \sum_{k=0}^{\infty}\frac{t^k}{k!}D^k$$

for $\operatorname{Re} t \geq 0$ with the operator norm $e^{-\frac{m-1}{2}\operatorname{Re} t}$ satisfying the composition law

$$e^{t_1 D}\circ e^{t_2 D} = e^{(t_1+t_2)D}.$$

Thus, e^{tD} $(\operatorname{Re} t > 0)$ forms a holomorphic one-parameter semigroup. Besides, the operator e^{tD} is self-adjoint if t is real, and is unitary if t is purely imaginary.

We ask:

Question. *Find an explicit formula of e^{tD}.*

In this context, our main results are stated as follows:

Theorem A (see Theorem 5.1.1). *The holomorphic semigroup* e^{tD} $(\operatorname{Re} t > 0)$ *is given by*

$$\big(e^{tD}u\big)(x) = \int_{\mathbb{R}^m} K^+(x,x';t)u(x')\frac{dx'}{|x'|} \quad \textit{for } u \in L^2(\mathbb{R}^m, \frac{dx}{|x|}),$$

where the integral kernel $K^+(x,x';t)$ *is defined by*

$$K^+(x,x';t) := \frac{2e^{-2(|x|+|x'|)\coth\frac{t}{2}}}{\pi^{\frac{m-1}{2}}\sinh^{m-1}\frac{t}{2}}\tilde{I}_{\frac{m-3}{2}}\Big(\frac{\psi(x,x')}{\sinh\frac{t}{2}}\Big).$$

Here, we set $\tilde{I}_\nu(z) := \left(\frac{z}{2}\right)^{-\nu} I_\nu(z)$ *(*I_ν *is the I-Bessel function (see* (8.5.2)*) and*

$$\psi(x,x') := 2\sqrt{2(|x||x'| + \langle x,x'\rangle)} = 4|x|^{\frac{1}{2}}|x'|^{\frac{1}{2}}\cos\frac{\theta}{2},$$

where $\theta \equiv \theta(x,x')$ *is the Euclidean angle between* x *and* x' *in* $\mathbb{R}^m$*.*

Particularly important is the special value at $t = \pi\sqrt{-1}$. We set

$$\begin{aligned} K^+(x,x') &:= \lim_{\varepsilon\downarrow 0} K^+(x,x';\pi\sqrt{-1}+\varepsilon) \\ &= \frac{2^{\frac{m-1}{2}}\psi(x,x')^{-\frac{m-3}{2}}}{\sqrt{-1}^{m-1}\pi^{\frac{m-1}{2}}} J_{\frac{m-3}{2}}(\psi(x,x')) \\ &= \frac{2}{\sqrt{-1}^{m-1}\pi^{\frac{m-1}{2}}}\tilde{J}_{\frac{m-3}{2}}(\psi(x,x')). \end{aligned}$$

Corollary B (see Theorem 6.1.1 and Corollary 6.2.1). *The unitary operator* $e^{\pi\sqrt{-1}D}$ *on* $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ *is given by the Hankel-type transform:*

$$T : L^2(\mathbb{R}^m, \frac{dx}{|x|}) \to L^2(\mathbb{R}^m, \frac{dx}{|x|}), \quad u \mapsto \int_{\mathbb{R}^m} K^+(x,x')u(x')\frac{dx'}{|x'|}.$$

This transform has the following properties:

(Inversion formula) $T^{-1} = (-1)^{m+1}T.$

(Plancherel formula) $\|Tu\|_{L^2(\mathbb{R}^m, \frac{dx}{|x|})} = \|u\|_{L^2(\mathbb{R}^m, \frac{dx}{|x|})}.$

Let us explain the backgrounds and motivation of our question from three different viewpoints:

(1) Hermite semigroup and its variants (see Subsection 1.2).

(2) $\mathfrak{sl}_2$-triple of differential operators on $\mathbb{R}^m$ (see Subsection 1.3).

(3) Minimal representation of the reductive group $O(m+1,2)$ (see Subsection 1.4).

1.2 Comparison with the Hermite operator $\mathcal{D}$

Let us compare our operator D on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ with the well-known operator $\mathcal{D}$ on $L^2(\mathbb{R}^m, dx)$ defined by

$$\mathcal{D} := \frac{1}{4}(\Delta - |x|^2). \tag{1.2.1}$$

We call this operator the *Hermite operator* following the terminology of R. Howe and E.-C. Tan [17]. Analogously to D, the Hermite operator $\mathcal{D}$ satisfies the following properties:

1) $\mathcal{D}$ extends to a self-adjoint operator on $L^2(\mathbb{R}^m, dx)$.

2) $\mathcal{D}$ has only discrete spectra $\{-\frac{1}{2}(j + \frac{m}{2}) : j = 0, 1, \cdots\}$ in $L^2(\mathbb{R}^m, dx)$.

We recall from [16, §5.3 and §6] that $\mathcal{D}$ gives rise to a holomorphic semigroup $e^{t\mathcal{D}}$ ($\operatorname{Re} t > 0$) (*Hermite semigroup*). Then the following results may be regarded as a prototype of Theorem A and Corollary B.

Fact C (see [16, §5] [31, §4.1]). *The holomorphic semigroup $e^{t\mathcal{D}}$ ($\operatorname{Re} t > 0$) is given by*

$$(e^{t\mathcal{D}}u)(x) = \int_{\mathbb{R}^m} \mathcal{K}(x, x'; t) u(x') dx',$$

where $\mathcal{K}$ is the Mehler kernel defined by

$$\mathcal{K}(x, x'; t) := \frac{1}{(2\pi \sinh \frac{t}{2})^{\frac{m}{2}}} \exp\Big(-\frac{1}{2}\, {}^t\!\begin{pmatrix} x \\ x' \end{pmatrix} A(t) \begin{pmatrix} x \\ x' \end{pmatrix}\Big). \tag{1.2.2}$$

Here, we set

$$A(t) := \frac{1}{\sinh \frac{t}{2}} \begin{pmatrix} \cosh \frac{t}{2} I_m & -I_m \\ -I_m & \cosh \frac{t}{2} I_m \end{pmatrix} \in GL(2m, \mathbb{R}).$$

In light of the limit formula:

$$\lim_{\varepsilon \downarrow 0} \mathcal{K}(x, x'; \pi\sqrt{-1} + \varepsilon) = \frac{1}{(2\pi\sqrt{-1})^{\frac{m}{2}}} e^{-\sqrt{-1}\langle x, x' \rangle},$$

the special value of the operator $e^{t\mathcal{D}}$ at $t = \pi\sqrt{-1}$ reduces to the (ordinary) Fourier transform:

Fact D. *The unitary operator $e^{\pi\sqrt{-1}\mathcal{D}}$ on $L^2(\mathbb{R}^m, dx)$ is nothing other than the Fourier transform $\mathcal{F}$:*

$$(\mathcal{F}f)(x) = \frac{1}{(2\pi\sqrt{-1})^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\sqrt{-1}\langle x, x' \rangle} f(x') dx'.$$

We shall see in Section 6 a group theoretic interpretation of the fact that T is unitary and $T^4 = \mathrm{id}$ as well as the fact that $\mathcal{F}$ is unitary and $\mathcal{F}^4 = \mathrm{id}$.

1.3 The action of $SL(2,\mathbb{R})^{\sim} \times O(m)$

The self-adjoint operator D defined by (1.1.1) arises in the context of the $\mathfrak{sl}_2$-triple of differential operators on $\mathbb{R}^m$ as follows. We define

$$\tilde{h} = 2\sum_{j=1}^{m} x_j \frac{\partial}{\partial x_j} + m - 1, \quad \tilde{e} = 2\sqrt{-1}|x|, \quad \tilde{f} = \frac{\sqrt{-1}}{2}|x|\Delta. \tag{1.3.1}$$

These operators $\tilde{h}, \tilde{e}$ and $\tilde{f}$ are skew self-adjoint operators on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, and satisfy the $\mathfrak{sl}_2$-relation:

$$[\tilde{h}, \tilde{e}] = 2\tilde{e}, \quad [\tilde{h}, \tilde{f}] = -2\tilde{f}, \quad [\tilde{e}, \tilde{f}] = \tilde{h}.$$

The operator D has the following expression

$$D = \frac{1}{2\sqrt{-1}}(-\tilde{e} + \tilde{f}), \tag{1.3.2}$$

which means that $\sqrt{-1}D$ corresponds to a generator of $\mathfrak{so}(2)$ in $\mathfrak{sl}(2,\mathbb{R})$. The $\mathfrak{sl}_2$-module $C_0^\infty(\mathbb{R}^m \setminus \{0\})$ exponentiates to a unitary representation of $SL(2,\mathbb{R})$ on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ (see Subsection 2.3 and Lemma 3.4.1).

On the other hand, there is a natural unitary representation of the orthogonal group $O(m)$ on the same space $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, and the actions of $SL(2,\mathbb{R})$ and $O(m)$ mutually commute. Then we have the following discrete and multiplicity-free decomposition into irreducible representations of $SL(2,\mathbb{R}) \times O(m)$ (see [22, Theorem A]):

$$L^2(\mathbb{R}^m, \frac{dx}{|x|}) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi_{2j+m-1}^{SL(2,\mathbb{R})} \otimes \mathcal{H}^j(\mathbb{R}^m).$$

Here, $\mathcal{H}^j(\mathbb{R}^m)$ denotes the space of harmonic polynomials on $\mathbb{R}^m$ of degree j, and $\pi_b^{SL(2,\mathbb{R})}$ stands for the irreducible unitary lowest weight representation of $SL(2,\mathbb{R})$ with minimal K-type $\mathbb{C}_b$ for $b \in \mathbb{N}_+ = \{1, 2, \cdots\}$. It is the limit of discrete series if $b = 1$, and holomorphic discrete series if $b \geq 2$.

In contrast, the Hermite operator $\mathcal{D}$ (see (1.2.1)) arises from the following $\mathfrak{sl}_2$-triple:

$$\tilde{h}' := \sum_{j=1}^{m} x_j \frac{\partial}{\partial x_j} + \frac{m}{2}, \quad \tilde{e}' := \frac{\sqrt{-1}}{2}|x|^2, \quad \tilde{f}' := \frac{\sqrt{-1}}{2}\Delta, \tag{1.3.3}$$

where the Hermite operator $\mathcal{D}$ is given by

$$\mathcal{D} = \frac{1}{2\sqrt{-1}}(-\tilde{e}' + \tilde{f}'). \tag{1.3.4}$$

This $\mathfrak{sl}_2$-triple also gives rise to the commutative actions of the double covering group $SL(2,\mathbb{R})^\sim$ of $SL(2,\mathbb{R})$ and $O(m)$ on $L^2(\mathbb{R}^m)$, whose irreducible decomposition amounts to (see [17, Chapter III, Theorem 2.4.4])

$$L^2(\mathbb{R}^m, dx) \simeq \sum_{j=0}^{\infty}{}^{\oplus} \pi_{j+\frac{m}{2}}^{SL(2,\mathbb{R})^\sim} \otimes \mathcal{H}^j(\mathbb{R}^m).$$

Here, $\pi_b^{SL(2,\mathbb{R})^\sim}$ stands for the irreducible unitary lowest weight representation of $SL(2,\mathbb{R})^\sim$ with minimal K-type $\mathbb{C}_b$ for $b \in \frac{1}{2}\mathbb{N}_+ = \{\frac{1}{2}, 1, \frac{3}{2}, \cdots\}$. It is the Weil representation if $b = \frac{1}{2}$, and is obtained by the representation of $SL(2,\mathbb{R})$ if $b \in \mathbb{N}_+$.

1.4 Minimal representation as hidden symmetry

The representation of $SL(2,\mathbb{R}) \times O(m)$ on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ in Subsection 1.3 extends to the irreducible unitary representation π_+ of the double covering group $G := SO_0(m+1,2)^\sim$ of the indefinite orthogonal group (see Subsections 2.3 and 3.1). If m is odd, this representation is well-defined also as a representation of $SO_0(m+1,2)$.

Similarly, the representation of $SL(2,\mathbb{R})^\sim \times O(m)$ on $L^2(\mathbb{R}^m, dx)$ extends to the unitary representation ϖ of the metaplectic group $G' = Sp(m,\mathbb{R})^\sim$.

These groups G and G' may be interpreted as *hidden symmetry* of $SL(2,\mathbb{R})^\sim \times O(m)$. Conversely, the group $SL(2,\mathbb{R})^\sim \times O(m)$ forms a 'dual pair' in each of the groups G and G'.

The unitary representations π_+ and ϖ are typical examples of 'minimal representations' of reductive Lie groups in the sense that the Gelfand–Kirillov dimension attains its minimum among infinite dimensional unitary representations or in the sense that its annihilator is the Joseph ideal in the enveloping algebra.

The unitary representation π_+ may be interpreted as the mass-zero spin-zero wave equation, or as the bound states of the Hydrogen atom (in m space dimensions), while the representation ϖ is sometimes referred to as the oscillator representation or as the (Segal–Shale–)Weil representation.

We shall review the L^2-realization of the minimal representation π_+ of $SO_0(m+1,2)^\sim$, that is, the analog of the Schrödinger model on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ in Subsection 3.1. See also [8, 16] for a nice introduction to the original Schrödinger model of the Weil representation of $Sp(m,\mathbb{R})^\sim$ on $L^2(\mathbb{R}^m)$.

To be more precise, we take

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := [e,f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to be a basis of $\mathfrak{sl}(2,\mathbb{R})$ and define injective Lie algebra homomorphisms

$$\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{so}(m+1,2),$$
$$\varphi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sp}(m,\mathbb{R})$$

(see Subsection 3.3) such that the differential operators (1.3.1) and (1.3.3) are obtained via ϕ and φ, respectively, that is,

$$d\pi_+(\phi(X)) = \tilde{X}, \quad d\varpi(\varphi(X)) = \tilde{X}'$$

holds for $X = e, f, h$. Next we set

$$z := \frac{1}{2\sqrt{-1}}(-e+f) = \frac{1}{2\sqrt{-1}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \sqrt{-1}\mathfrak{sl}(2,\mathbb{R}).$$

Then $e^{\sqrt{-1}\mathbb{R}\phi(z)}$ is the center of the maximal compact subgroup of G for $m > 1$, while $e^{\sqrt{-1}\mathbb{R}\varphi(z)}$ is that of G' (we use the same notations ϕ and φ for their complex linear extensions).

In this context, we shall see in Lemma 3.4.3 and Remark 3.4.5 that the differential operators D and $\mathcal{D}$ are given by

$$D = d\pi_+(\phi(z)), \qquad \mathcal{D} = d\varpi(\varphi(z)). \tag{1.4.1}$$

Thanks to these formulas, Theorem A and Corollary B are also useful in the analysis on minimal representations of $SO_0(m+1,2)^\sim$ as well as $Sp(m,\mathbb{R})^\sim$ in the following contexts:

1) **The Gelfand–Gindikin program — Theorem A**.

The *Gelfand–Gindikin program* asks for extending a given unitary representation of a real semisimple Lie group G to a holomorphic object of some complex submanifold in its complexification $G_{\mathbb{C}}$. Stanton and Olshanskiĭ [29, 27] independently gave a general framework of the Gelfand–Gindikin program for holomorphic discrete series. Their abstract results are enriched, for example for $G' = Sp(m,\mathbb{R})^\sim$, by the explicit formula of the Hermite semigroup $e^{t\mathcal{D}} = \varpi(e^{t\varphi(z)})$ for the Weil representation ϖ on $L^2(\mathbb{R}^m)$ by Howe [16]. Likewise, Theorem A gives an explicit formula of the semigroup $e^{tD} = \pi_+(e^{t\phi(z)})$ for the minimal representation of $G = SO_0(m+1,2)^\sim$. Since $e^{t\phi(z)} \in G_{\mathbb{C}} \setminus G$ for $\operatorname{Re} t > 0$, Theorem A can be interpreted as a descendent of the Gelfand–Gindikin program.

2) **The unitary inversion operator — Corollary B**.

In the Schrödinger model of the minimal representation, G acts only on the function space $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ but does not act on the underlying geometry

$\mathbb{R}^m$ itself. One may observe this fact by the aforementioned formula $\tilde{f} = d\pi_+(\phi(f))$, which does not act on functions on $\mathbb{R}^m$ as a vector field but acts as a differential operator of second order on $\mathbb{R}^m$ (see (1.3.1)). To see how G acts on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, we use the facts that

1) G is generated by $\overline{P^{\max}}$ and w_0.

2) The $\overline{P^{\max}}$ action on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ is easily described (see Subsection 2.2).

Here, $\overline{P^{\max}}$ is a maximal parabolic subgroup of G (see Subsection 2.1) and $w_0 := e^{\pi\sqrt{-1}Z} \in G$ sends $\overline{P^{\max}}$ to its opposite parabolic subgroup $P^{\max}$. Geometrically, $\overline{P^{\max}}$ is essentially the conformal affine transformation group on the flat standard Lorentz manifold $\mathbb{R}^{m,1}$ (the *Minkowski space*), and $w_0 := e^{\pi\sqrt{-1}Z} \in G$ acts on $\mathbb{R}^{m,1}$ as the 'inversion' element (see Subsection 6.1).

Thus the representation π_+ of G on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ would be understood if we find an explicit formula for $\pi_+(w_0)$. But since the formula $w_0 = e^{\pi\sqrt{-1}\phi(z)}$ implies $\pi_+(w_0) = e^{\pi\sqrt{-1}D}$, Corollary B answers this question. This parallels the fact that the Weil representation is generated by the (natural) action of the Siegel parabolic subgroup P_{Siegel} and the Fourier transform $\mathcal{F} = e^{\pi\sqrt{-1}\mathcal{D}}$ (see Fact D).

Briefly, we pin down the analogy in the table below. Howe [16] established the left-hand side of the table for the oscillator representation ϖ of $Sp(m,\mathbb{R})^\sim$, while Theorem A and Corollary B supply the right-hand side of the table for the minimal representation π_+ of $SO_0(m+1,2)^\sim$.

	$\mathfrak{sp}(m,\mathbb{R})$	$\mathfrak{so}(m+1,2)$
minimal representation	$(d\varpi, L^2(\mathbb{R}^m))$	$(d\pi_+, L^2(\mathbb{R}^m, \frac{dx}{\lvert x\rvert}))$
e	$\frac{\sqrt{-1}}{2}\lvert x\rvert^2$	$2\sqrt{-1}\lvert x\rvert$
f	$\frac{\sqrt{-1}}{2}\Delta$	$\frac{\sqrt{-1}}{2}\lvert x\rvert\Delta$
h	$E_x + \frac{m}{2}$	$2E_x + m - 1$
$z = \frac{-e+f}{2\sqrt{-1}}$	$\mathcal{D} := \frac{1}{4}(\Delta - \lvert x\rvert^2)$	$D := \frac{1}{4}\lvert x\rvert\Delta - \lvert x\rvert$
holomorphic semigroup e^{tz}	$\mathcal{K}(x,x';t)$	$K(x,x';t)$
inversion $e^{\pi\sqrt{-1}z}$	Fourier transform	Hankel transform
maximal parabolic subgroup	P_{Siegel}	$\overline{P^{\max}}$

Analogous results to Corollary B were previously known for some singular unitary highest weight representations. For example, see a paper [4] by Ding, Gross, Kunze and Richards for those of $U(n,n)$. Since $SU(2,2)$ is a double covering group of $SO_0(4,2)$, Corollary B in the case $m=3$ essentially

corresponds to [4, Corollary 7.5] in the case $(n, k) = (2, 1)$ in their notation. However, our proof based on an analytic continuation (see Theorem A) is different from theirs. Also in [20], we shall find the inversion operator $\pi(w_0)$ for the minimal representation of $O(p, q)$ for $p + q$ even, and in particular give yet another proof of Corollary B for odd m.

We also present explicit integral formulas of $\pi(e^{tZ})$ and $\pi(w_0)$ when restricted to radial functions and alike (see Theorem 4.1.1). This yields a group theoretic interpretation of some classic formulas of special functions of one variable including Weber's second exponential integral formula on Bessel function and the reciprocal and the Parseval-Plancherel formula for the Hankel transform.

This article is organized as follows. After summarizing the preliminary results on the L^2-model of the minimal representation π, we find explicitly which function arises for describing K-types in $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, and define a holomorphic extension $\pi_+(e^{tZ})$ in Section 3. The integral formula of the 'radial' part of $\pi_+(e^{tZ})$ is given in Theorem 4.1.1. Theorem A is proved in Section 5 by using the result of Section 4. Taking the special value at $t = \pi\sqrt{-1}$, we obtain the integral formula of $\pi(w_0)$ corresponding to the inversion element w_0. This corresponds to Corollary B, and is proved in Section 6. Our integral formula for $\pi(w_0)$ enables us to write explicitly the action of the whole group $G = SO_0(m+1, 2)\tilde{}$ on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$. This is given in Section 7. For the convenience of the reader, we collect basic formulas of special functions in a way that we use in this article.

The main results of the paper were announced in [19] with a sketch of the proof.

Notation: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}_+ = \{1, 2, 3, \dots\}$, $\mathbb{R}_{\leq 0} = \{x \in \mathbb{R} : x \leq 0\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

2 Preliminary results on the minimal representation of $O(m+1, 2)$

This section gives a brief review on the known results of the L^2-model of the minimal representation of $O(m + 1, 2)$ in a way that we shall use later. We shall give an explicit action of the maximal parabolic subgroup $\overline{P^{\max}}$ and the Lie algebra $\mathfrak{g}$. Furthermore, we state an explicit K-type decomposition of $L^2(C_+)$ even though the action of K itself is not given explicitly here (see Section 7 for this).

2.1 Maximal parabolic subgroup of the conformal group

Let $O(m+1,2)$ be the indefinite orthogonal group which preserves the quadratic form $x_0^2+\cdots+x_m^2-x_{m+1}^2-x_{m+2}^2$ of signature $(m+1,2)$. We denote by $e_0,\cdots,e_{m+2}$ the standard basis of $\mathbb{R}^{m+3}$, and by E_{ij} $(0\le i,j\le m+2)$ the matrix unit. We set

$$\varepsilon_j := \begin{cases} 1 & (1\le j\le m), \\ -1 & (j=m+1). \end{cases}$$

We take the following elements of the Lie algebra $\mathfrak{o}(m+1,2)$:

$$\begin{aligned} \overline{N}_j &:= E_{j,0}+E_{j,m+2}-\varepsilon_j E_{0,j}+\varepsilon_j E_{m+2,j} \quad (1\le j\le m+1), \\ N_j &:= E_{j,0}-E_{j,m+2}-\varepsilon_j E_{0,j}-\varepsilon_j E_{m+2,j} \quad (1\le j\le m+1), \\ E &:= E_{0,m+2}+E_{m+2,0}, \end{aligned} \tag{2.1.1}$$

and define subalgebras of $\mathfrak{o}(m+1,2)$ by

$$\overline{\mathfrak{n}^{\max}} := \sum_{j=1}^{m+1}\mathbb{R}\overline{N}_j, \quad \mathfrak{n}^{\max} := \sum_{j=1}^{m+1}\mathbb{R}N_j, \quad \mathfrak{a} := \mathbb{R}E.$$

Then we define the following subgroups of $O(m+1,2)$:

$$\begin{aligned} M_+^{\max} &:= \{g\in O(m+1,2) : g\cdot e_0 = e_0,\ g\cdot e_{m+2}=e_{m+2}\} \\ &\simeq\ O(m,1), \\ M^{\max} &:= M_+^{\max}\cup\{-I_{m+3}\}\cdot M_+^{\max} \\ &\simeq\ O(m,1)\times\mathbb{Z}_2, \\ \overline{N^{\max}} &:= \exp(\overline{\mathfrak{n}^{\max}}), \\ N^{\max} &:= \exp(\mathfrak{n}^{\max}), \\ A &:= \exp(\mathfrak{a}). \end{aligned}$$

For $b=(b_1,\ldots,b_{m+1})\in\mathbb{R}^{m+1}$, we set

$$\begin{aligned} \overline{n}_b :=&\ \exp(\sum_{j=1}^{m+1} b_j\overline{N}_j) \qquad\qquad (2.1.2) \\ =&\ I_{m+3}+\sum_{j=1}^{m+1} b_j\overline{N}_j+\frac{Q(b)}{2}(-E_{0,0}-E_{0,m+1}+E_{m+1,0}+E_{m+1,m+1}), \end{aligned}$$

where $Q(b)$ is the quadratic form of signature $(m, 1)$ given by

$$Q(b) := b_1^2 + \cdots + b_m^2 - b_{m+1}^2.$$

The Lie group $\overline{N^{\max}}$ is abelian, and we have an isomorphism of Lie groups:

$$\mathbb{R}^{m+1} \xrightarrow{\sim} \overline{N^{\max}}, \quad b \mapsto \overline{n}_b.$$

It is readily seen from (2.1.2) that

$$\overline{n}_b(e_0 - e_{m+2}) = e_0 - e_{m+2}, \tag{2.1.3}$$

$$\overline{n}_b(e_0 + e_{m+2}) = {}^t\!(1 - Q(b), 2b_1, \ldots, 2b_{m+1}, 1 + Q(b)). \tag{2.1.4}$$

We also note

$$e^{tE}(e_0 + e_{m+2}) = e^t(e_0 + e_{m+2}). \tag{2.1.5}$$

The subgroup $M_+^{\max}\overline{N^{\max}}$ is isomorphic to the semidirect product group $O(m, 1) \ltimes \mathbb{R}^{m+1}$ via the bijection $\mathbb{R}^{m+1} \xrightarrow{\sim} \overline{N^{\max}}$, $b \mapsto \overline{n}_b$. In this context, $M_+^{\max}\overline{N^{\max}}$ is regarded as the group of isometries of the Minkowski space $\mathbb{R}^{m,1}$, while $O(m + 1, 2)$ is the group of Möbius transformations on $\mathbb{R}^{m,1}$ preserving its conformal structure.

Next, we define a maximal parabolic subgroup

$$\overline{P^{\max}} := M^{\max}A\overline{N^{\max}}.$$

In our analysis of the minimal representation of $O(m+1, 2)$, $\overline{P^{\max}}$ plays an analogous role to the Siegel parabolic subgroup of the metaplectic group $Sp(m, \mathbb{R})\widetilde{\ }$ for the Weil representation.

2.2 L^2-model of the minimal representation

We shall briefly review the L^2-model of the minimal representation of $O(m+1, 2)$. Let $C_\pm$ be the forward and the backward light cone respectively:

$$C_\pm := \{(\zeta_1, \cdots, \zeta_{m+1}) \in \mathbb{R}^{m+1} : \pm\zeta_{m+1} > 0, \quad \zeta_1^2 + \cdots + \zeta_m^2 = \zeta_{m+1}^2\},$$

and C be its disjoint union $C_+ \cup C_+$, that is, C is the conical subvariety with respect to the quadratic form of signature $(m, 1)$:

$$C = \{\zeta \in \mathbb{R}^{m+1} \setminus \{0\} : Q(\zeta) = 0\}. \tag{2.2.1}$$

Note that $M_+^{\max}$ acts on C transitively.

The measure $d\mu$ on C is naturally defined to be $\delta(Q)$, the generalized function associated to the quadratic form Q (see [11, Chapter III, §2]).

Then we form a unitary representation π of $\overline{P^{\max}}$ on the Hilbert space $L^2(C, d\mu) \equiv L^2(C)$ as follows: for $\psi \in L^2(C)$,

$$(\pi(e^{tE})\psi)(\zeta) := e^{-\frac{m-1}{2}t}\psi(e^{-t}\zeta) \qquad (t \in \mathbb{R}), \tag{2.2.2}$$

$$(\pi(m)\psi)(\zeta) := \psi({}^t m\zeta) \qquad (m \in M_+^{\max}), \tag{2.2.3}$$

$$(\pi(-I_{m+3})\psi)(\zeta) := (-1)^{\frac{m-1}{2}}\psi(\zeta), \tag{2.2.4}$$

$$(\pi(\overline{n}_b)\psi)(\zeta) := e^{2\sqrt{-1}(b_1\zeta_1+\cdots+b_{m+1}\zeta_{m+1})}\psi(\zeta) \qquad (b \in \mathbb{R}^{m+1}). \tag{2.2.5}$$

Then π is irreducible and unitary as a $\overline{P^{\max}}$-module, and it is proved in [23, Theorem 4.9] that the $\overline{P^{\max}}$-module $(\pi, L^2(C))$ extends to an irreducible unitary representation of $O(m+1, 2)$ if m is odd. We shall denote this representation of $O(m+1, 2)$ by the same letter π. The direct sum decomposition

$$L^2(C) = L^2(C_+) \oplus L^2(C_-) \tag{2.2.6}$$

yields a branching law $\pi = \pi_+ \oplus \pi_-$ with respect to the restriction $O(m+1, 2) \downarrow SO_0(m+1, 2)$, where $SO_0(m+1, 2)$ is the identity component of $O(m+1, 2)$. The irreducible representations π_+ and π_- of $SO_0(m+1, 2)$ are contragredient to each other, one is a highest weight module, and the other is a lowest weight module.

2.3 K-type decomposition

Let $SO(2)\tilde{}$ be the double covering of $SO(2)$. We write η for the unique element of $SO(2)\tilde{}$ of order two. Then, we have an exact sequence:

$$1 \to \{1, \eta\} \to SO(2)\tilde{} \to SO(2) \to 1.$$

Let

$$G = SO_0(m+1, 2)\tilde{} \tag{2.3.1}$$

be the double covering group of $SO_0(m+1, 2)$ characterized as follows: a maximal compact subgroup K is of the form $K_1 \times K_2 \simeq SO(m+1) \times SO(2)\tilde{}$ and the kernel of the covering map $G \to SO_0(m+1, 2)$ is given by $\{(1, 1), (1, \eta)\}$. Likewise, the double covering group $O(m+1, 2)\tilde{}$ of $O(m+1, 2)$ is defined.

If m is odd, the irreducible representation $(\pi_\pm, L^2(C_\pm))$ defined in Subsection 2.2 extends to that of $SO_0(m+1, 2)$, and therefore, also that of $G = SO_0(m+1, 2)\tilde{}$ as we proved it more generally for $O(p, q)$ ($p+q$: even) in [23]. We shall use the same letter π_+ to denote the extension to

$SO_0(m+1,2)$ or G. If m is even, by [28], the irreducible unitary representation $(\pi_\pm, L^2(C_\pm))$ is still well-defined as a representation of G, whose Lie algebra $\mathfrak{g} = \mathfrak{so}(m+1,2)$ of G acts in the same manner as in the case of odd m (see Subsection 2.4).

The K-type formula of $(\pi_\pm, L^2(C_\pm))$ is given as follows:

$$L^2(C_\pm)_K \simeq \bigoplus_{a=0}^{\infty} \mathcal{H}^a(\mathbb{R}^{m+1}) \boxtimes \mathbb{C}e^{\pm(a+\frac{m-1}{2})\sqrt{-1}\theta}. \tag{2.3.2}$$

(For example, this formula can be read from [28, §1.3] by substituting $d = m-1, p = 1$ and $q = 0$.) Here, $\mathcal{H}^a(\mathbb{R}^{m+1})$ stands for the representation of $SO(m+1)$ on the space of the spherical harmonics which is irreducible if $m > 1$ (see Subsection 8.4).

Likewise, the representation $(\pi, L^2(C))$ of $O(m+1,2)^{\sim}$ decomposes when restricted to its maximal compact subgroup as follows (see [21, Theorem 3.6.1]):

$$L^2(C)_K \simeq \bigoplus_{a=0}^{\infty} \mathcal{H}^a(\mathbb{R}^{m+1}) \boxtimes \mathcal{H}^{a+\frac{m-1}{2}}(\mathbb{R}^2). \tag{2.3.3}$$

$\mathcal{H}^{a+\frac{m-1}{2}}(\mathbb{R}^2)$ decomposes into $\mathbb{C}e^{(a+\frac{m-1}{2})\sqrt{-1}\theta} \oplus \mathbb{C}e^{-(a+\frac{m-1}{2})\sqrt{-1}\theta}$ as $SO(2)$-modules (see Subsection 8.4). This corresponds to the decomposition

$$L^2(C) = L^2(C_+) \oplus L^2(C_-),$$

for which the K-type formula is given by (2.3.2).

2.4 Infinitesimal action of the minimal representation

For $\overline{N}_j, N_j (1 \le j \le m+1)$ and E, we define linear transformations on the space $\mathcal{S}'(\mathbb{R}^{m+1})$ of tempered distributions by

$$d\hat{\varpi}(\overline{N}_j) := 2\sqrt{-1}\zeta_j, \tag{2.4.1}$$

$$d\hat{\varpi}(N_j) := \sqrt{-1}\left(-\frac{m+3}{2}\varepsilon_j\frac{\partial}{\partial\zeta_j} - E_\zeta\varepsilon_j\frac{\partial}{\partial\zeta_j} + \frac{1}{2}\zeta_j\Box_\zeta\right), \tag{2.4.2}$$

$$d\hat{\varpi}(E) := -\frac{m+3}{2} - E_\zeta, \tag{2.4.3}$$

where we set

$$\Box_\zeta := \frac{\partial^2}{\partial\zeta_1^2} + \frac{\partial^2}{\partial\zeta_2^2} + \cdots + \frac{\partial^2}{\partial\zeta_m^2} - \frac{\partial^2}{\partial\zeta_{m+1}^2}, \quad E_\zeta := \sum_{j=1}^{m+1}\zeta_j\frac{\partial}{\partial\zeta_j}.$$

Then, we recall from [23] that this generates the infinitesimal action $d\pi$ of the Lie algebra $\mathfrak{so}(m+1,2)$, and we have the following commutative diagram for any $X \in \mathfrak{so}(m+1,2)$:

$$\begin{array}{ccc} L^2(C)_K & \xrightarrow{\imath} & \mathcal{S}'(\mathbb{R}^{m+1}) \\ d\pi(X) \Big\downarrow & & \Big\downarrow d\hat{\varpi}(X) \\ L^2(C)_K & \xrightarrow{\imath} & \mathcal{S}'(\mathbb{R}^{m+1}). \end{array} \tag{2.4.4}$$

Here, $\imath : L^2(C) \to \mathcal{S}'(\mathbb{R}^{m+1})$ is given by $u(\zeta) \mapsto u(\zeta)\delta(Q)$. This is well-defined and injective if $m > 1$ (see [23, §3.4]).

3 Branching law of π_+

The main goal of this section is Proposition 3.2.1, which explicitly describes special functions that arise as K-types in the 'Schrödinger model' on $L^2(\mathbb{R}^m, |x|^{-1}dx)$ of the minimal representation of the double covering group G of $SO_0(m+1,2)$.

3.1 Schrödinger model of the minimal representation

We have used the variables $\zeta = (\zeta_1, \cdots, \zeta_{m+1})$ for the positive cone $C_+ \subset \mathbb{R}^{m+1}$ in Section 2, and will use the letter $x = (x_1, \cdots, x_m)$ for the coordinate of $\mathbb{R}^m$. The projection

$$p : \mathbb{R}^{m+1} \to \mathbb{R}^m, \quad (\zeta_1, \cdots, \zeta_m, \zeta_{m+1}) \mapsto (\zeta_1, \cdots, \zeta_m). \tag{3.1.1}$$

induces a diffeomorphism from C_+ onto $\mathbb{R}^m \setminus \{0\}$, and the measure $d\mu$ on C_+ is given by $\delta(Q)$, and therefore is pushed forward to $\frac{1}{2|x|}dx = \frac{1}{2|x|}dx_1 \cdots dx_m$. Thus, we have a unitary isomorphism:

$$\sqrt{2}p^* : L^2(\mathbb{R}^m, \frac{dx}{|x|}) \xrightarrow{\sim} L^2(C_+). \tag{3.1.2}$$

Through this isomorphism, we can realize the minimal representation of G on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ as well. Named after the Schrödinger model of the Weil representation, we say this model is the *Schrödinger model* of the minimal representation of G. We shall work with this model from now on.

3.2 K-finite functions on the forward light cone C_+

This section refines the K-type decomposition (2.3.2) by providing an explicit irreducible decomposition:

$$L^2(\mathbb{R}^m, \frac{dx}{|x|})_K = \bigoplus_{a=0}^{\infty} W_a = \bigoplus_{a=0}^{\infty} \bigoplus_{l=0}^{a} W_{a,l} \tag{3.2.1}$$

according to the following chain of subgroups:

$$\begin{array}{ccccc} G & \supset & K & \supset & R := K \cap (M_+^{\max})_0 \\ \shortparallel & & \shortparallel & & \shortparallel \\ SO_0(m+1,2)\tilde{} & \supset & SO(m+1) \times SO(2)\tilde{} & \supset & SO(m). \end{array}$$

Here, the K-irreducible subspace W_a of $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ and the R-irreducible supspace $W_{a,l}$ of W_a is characterized by

$$W_a \simeq \mathcal{H}^a(\mathbb{R}^{m+1}) \boxtimes \mathbb{C}e^{(a+\frac{m-1}{2})\theta} \quad (\text{see } (2.3.2)), \tag{3.2.2}$$

$$W_{a,l} \simeq \mathcal{H}^l(\mathbb{R}^m), \tag{3.2.3}$$

as K-modules and as R-modules, respectively. Here, we note that the $SO(m)$-module $\mathcal{H}^l(\mathbb{R}^m)$ occurs exactly once in the $O(m+1)$-module $\mathcal{H}^a(\mathbb{R}^{m+1})$ if $0 \le l \le a$ (see Subsection 8.4 (4)).

Proposition 3.2.1 will describe the subspace $W_{a,l}$ of $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ by means of Laguerre polynomials $L_n^{\alpha}(x)$ (see (8.1.1) for the definition). For this, we set

$$f_{a,l}(r) := L_{a-l}^{m-2+2l}(4r) r^l e^{-2r} \quad (0 \le l \le a), \tag{3.2.4}$$

and define injective linear maps by

$$j_{a,l} : \mathcal{H}^l(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m), \quad \phi(\omega) \mapsto (j_{a,l}\phi)(r\omega) := f_{a,l}(r)\phi(\omega).$$

Here, we have identified $\mathbb{R}^m$ with $\mathbb{R}_+ \times S^{m-1}$ by the polar coordinate

$$\mathbb{R} \times S^{m-1} \to \mathbb{R}^m, \quad (r,\omega) \mapsto r\omega. \tag{3.2.5}$$

Then, we have:

Proposition 3.2.1. 1)

$$j_{a,l}\big(\mathcal{H}^l(\mathbb{R}^m)\big) \subset L^1(\mathbb{R}^m, \frac{dx}{|x|}) \cap L^2(\mathbb{R}^m, \frac{dx}{|x|}). \tag{3.2.6}$$

2) *Furthermore, the image of $j_{a,l}$ coincides with the R-type $W_{a,l}$:*

$$j_{a,l}\big(\mathcal{H}^l(\mathbb{R}^m)\big) = W_{a,l}.$$

Remark 3.2.2. *Proposition 3.2.1 (2) asserts in particular that* $j_{a,l}(\mathcal{H}^l(\mathbb{R}^m))$ *and* $j_{a',l}(\mathcal{H}^l(\mathbb{R}^m))$ *are orthogonal to each other if* $a \neq a'$ $(a, a' \geq l)$*. More than this, it gives a representation theoretic proof of the fact that* $\{f_{a,l}(r) : a = l, l+1, \cdots\}$ *forms a complete orthogonal basis of* $L^2(\mathbb{R}_+, r^{m-2}dr)$ *(see Lemma 4.3.1 for the normalization).*

Remark 3.2.3. *The indefinite orthogonal group* $O(p,q)$ $(p+q$ *: even,* $p, q \geq 2$*, and* $(p,q) \neq (2,2))$ *has a minimal representation* π *whose minimal* K*-type is of the form* $\mathcal{H}^0(\mathbb{R}^p) \otimes \mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)$ *if* $p \geq q$*. In the* L^2*-model of* π*, we have proved that any* $M^{\max}$*-fixed vector in* $\mathcal{H}^0(\mathbb{R}^p) \otimes \mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)$ *is a scalar multiple of the function* $r^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2r)$ *(see [23, Theorem 5.5] for a precise statement), where* $K_\nu(z)$ *denotes the* K*-Bessel function. Since* $K_{-\frac{1}{2}}(2r) = \frac{\sqrt{\pi}}{2} r^{-\frac{1}{2}} e^{-2r}$ *and* $L_0^\alpha(x) = 1$*, we have* $r^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2r) = \frac{\sqrt{\pi}}{2} e^{-2r} = \frac{\sqrt{\pi}}{2} f_{0,0}(r)$ *if* $q = 2$*. This vector is a generator of the one dimensional vector space* $W_{0,0}$*.*

Remark 3.2.4 (Weil representation). *Let us compare our representation on* $L^2(\mathbb{R}^m, |x|^{-1}dx)$ *with the (original) Schrödinger model on* $L^2(\mathbb{R}^m)$ *of the Weil representation of* $G' = Sp(m, \mathbb{R})^\sim$*. The counterpart to Proposition 3.2.1 can be stated as follows: we set for* $0 \leq l \leq a$

$$f'_{a,l}(r) := L_{a-l}^{\frac{m-2}{2}+l}(r^2) r^l e^{-\frac{r^2}{2}}, \tag{3.2.7}$$

and define linear maps by

$$j'_{a,l} : \mathcal{H}^l(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m), \quad \phi(\omega) \mapsto (j'_{a,l}\phi)(r\omega) = f'_{a,l}(r)\phi(\omega).$$

Then for any $\phi \in \mathcal{H}^l(\mathbb{R}^m)$*,* $j'_{a,l}(\phi)$ *is square integrable on* $\mathbb{R}^m$*, and its image* $j'_{a,l}(\mathcal{H}^l(\mathbb{R}^m))$ *is characterized by the following properties: Let* $(K', R') = (U(m)^\sim, O(m)^\sim)$*.*

1) $j'_{a,l}(\mathcal{H}^l(\mathbb{R}^m))$ *is isomorphic to* $\mathcal{H}^l(\mathbb{R}^m)$ *as* R'*-modules,*

2) it is contained in the K'*-type isomorphic to* $S^a(\mathbb{C}) \otimes \det^{\frac{1}{4}}$*.*

The remaining part of this section is organized as follows. In Subsection 3.3, we shall define a central element Z of (the complexification of) the Lie algebra $\mathfrak{k}$ of K, and compute its differential action $d\pi_+(Z)$ (see Lemma 3.4.3). By using this explicit form of $d\pi_+(Z)$, we prove Proposition 3.2.1 in Subsection 3.5. Finally, in Subsection 3.6, by looking at the eigenvalues of $d\pi_+(Z)$ (see Lemma 3.5.1), we shall see that $\{\pi_+(e^{tZ}) := \exp(td\pi_+(Z)) : \operatorname{Re} t > 0\}$ forms a holomorphic semigroup of contraction operators.

In Section 5, we shall find an explicit integral kernel of this semigroup by using Proposition 3.2.1.

3.3 Description of infinitesimal generators of $\mathfrak{sl}(2,\mathbb{R})$

Let

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := [e,f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

be the standard basis of $\mathfrak{sl}(2,\mathbb{R})$. With the notation (2.1.1), we define a Lie algebra homomorphism

$$\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{so}(m+1,2) \tag{3.3.1}$$

by

$$\phi(e) = \overline{N}_{m+1}, \quad \phi(f) = N_{m+1}, \quad \phi(h) = -2E. \tag{3.3.2}$$

In this subsection, we shall explicitly describe $d\pi_+(\phi(e)), d\pi_+(\phi(f))$, and $d\pi_+(\phi(h))$ as differential operators on $\mathbb{R}^m$.

Lemma 3.3.1. *Let $E_x = \sum_{j=1}^m x_j \frac{\partial}{\partial x_j}$ and $\Delta = \sum_{j=1}^m \frac{\partial^2}{\partial x_j^2}$. Then we have:*

$$d\pi_+(\phi(h)) = 2E_x + m - 1, \tag{3.3.3}$$

$$d\pi_+(\phi(e)) = 2\sqrt{-1}|x|, \tag{3.3.4}$$

$$d\pi_+(\phi(f)) = \frac{\sqrt{-1}}{2}|x|\Delta. \tag{3.3.5}$$

Remark 3.3.2. *Lemma 3.3.1 corresponds to an analogous result for the Schrödinger model of the Weil representation $(\varpi, L^2(\mathbb{R}^m))$ of $Sp(m,\mathbb{R})^\sim$ as follows: by the matrix realization of the real symplectic Lie algebra $\mathfrak{sp}(m,\mathbb{R})$, we define a Lie algebra homomorphism $\varphi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sp}(m,\mathbb{R})$ by*

$$\varphi(e) = \begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix}, \quad \varphi(f) = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix}, \quad \varphi(h) = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}. \tag{3.3.6}$$

Then, $\{d\varpi(\varphi(h)), d\varpi(\varphi(f)), d\varpi(\varphi(f))\}$ is no other than the $\mathfrak{sl}_2$-triple of differential operators $\{\tilde{h}', \tilde{e}', \tilde{f}'\}$ on $\mathbb{R}^m$ given in (1.3.3).

Proof of Lemma 3.3.1. First we compute $d\pi_+(\phi(h)) = d\pi_+(-2E)$. For this, we use the formula of $d\hat{\varpi}$ on $\mathbb{R}^{m+1}$ in Subsection 2.4, and then compute the formula of $d\pi_+$ on the positive cone C_+ (or on the coordinate space $\mathbb{R}^m$) through the embedding $\imath : L^2(C) \to \mathcal{S}'(\mathbb{R}^{m+1})$, $u(\zeta) \mapsto u(\zeta)\delta(Q)$. Since the distribution $\delta(Q)$ is homogeneous of degree -2, we note that $E_\zeta \delta(Q) = -2\delta(Q)$. Therefore,

$$\begin{aligned}(d\pi_+(-2E)u)\delta(Q) =& -2d\hat{\varpi}(E)(u\delta(Q)) \quad \text{by (2.4.4)} \\ =& (2E_\zeta + m + 3)(u\delta(Q)) \quad \text{by (2.4.3)} \\ =& (2E_\zeta + m - 1)u \cdot \delta(Q).\end{aligned}$$

Now by identifying $L^2(C_+)$ with $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ by (3.1.1), we obtain (3.3.3).

The second formula (3.3.4) follows immediately from (2.4.1).

We shall show the third formula (3.3.5). In light of $\phi(f) = N_{m+1}$ (see (3.3.2)), by (2.4.2), we have

$$d\hat{\varpi}(\phi(f)) = \sqrt{-1}\Big(\frac{m+3}{2}\frac{\partial}{\partial \zeta_{m+1}} + E_\zeta \frac{\partial}{\partial \zeta_{m+1}} + \frac{\zeta_{m+1}}{2}\Box_\zeta\Big). \tag{3.3.7}$$

In order to compute the action $d\hat{\varpi}(\phi(f))$ along the cone C_+, we use the following coordinate on $\mathbb{R}^{m+1}$:

$$\mathbb{R} \times \mathbb{R}_+ \times S^{m-1} \to \mathbb{R}^{m+1}, \quad (Q, r, \omega) \mapsto (r\omega, \sqrt{r^2 - Q}). \tag{3.3.8}$$

Claim 3.3.3. *With the above coordinate, the differential operator $d\hat{\varpi}(\phi(f))$ on $\mathcal{S}'(\mathbb{R}^{m+1})$ takes the form:*

$$d\hat{\varpi}(\phi(f)) = \sqrt{-1}\sqrt{r^2 - Q}\Big(\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{m-1}{2r}\frac{\partial}{\partial r} + \frac{\Delta_{S^{m-1}}}{2r^2} - 2Q\frac{\partial^2}{\partial Q^2} - 4\frac{\partial}{\partial Q}\Big).$$

Proof of Claim 3.3.3. We start with a new coordinate $\mathbb{R}^{m+1} = \mathbb{R}^m \oplus \mathbb{R}$ by

$$\mathbb{R}_+ \times S^{m-1} \times \mathbb{R} \to \mathbb{R}^{m+1}, \quad (R, \omega, \zeta_{m+1}) \mapsto (R\omega, \zeta_{m+1}). \tag{3.3.9}$$

Then, clearly,

$$\begin{aligned} E_\zeta =& R\frac{\partial}{\partial R} + \zeta_{m+1}\frac{\partial}{\partial \zeta_{m+1}} \\ \Box_\zeta =& \Delta_{\mathbb{R}^m} - \frac{\partial^2}{\partial \zeta_{m+1}^2} = \Big(\frac{\partial^2}{\partial R^2} + \frac{m-1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\Delta_{S^{m-1}}\Big) - \frac{\partial^2}{\partial \zeta_{m+1}^2}. \end{aligned}$$

The coordinate (3.3.8) is obtained by the composition of (3.3.9) and

$$r = R, \quad Q = R^2 - \zeta_{m+1}^2.$$

In light of

$$\frac{\partial}{\partial R} = \frac{\partial}{\partial r} + 2r\frac{\partial}{\partial Q}, \quad \frac{\partial}{\partial \zeta_{m+1}} = -2\sqrt{r^2 - Q}\frac{\partial}{\partial Q}, \tag{3.3.10}$$

we get

$$E_\zeta = r\frac{\partial}{\partial r} + 2Q\frac{\partial}{\partial Q}, \tag{3.3.11}$$

$$\Box_\zeta = \frac{\partial^2}{\partial r^2} + 4Q\frac{\partial^2}{\partial Q^2} + 4r\frac{\partial^2}{\partial r \partial Q} + 2(m+1)\frac{\partial}{\partial Q} + \frac{m-1}{r}\frac{\partial}{\partial r} + \frac{\Delta_{S^{m-1}}}{r^2}. \tag{3.3.12}$$

Now substituting (3.3.10), (3.3.11) and (3.3.12) into (3.3.7), we get the claim. $\square$

Given $u \in L^2(C_+)_K$, we extend it to a distribution $\tilde{u} \equiv \tilde{u}(Q, r, \omega) \in \mathcal{S}'(\mathbb{R}^{m+1}) \cap C^\infty(\mathbb{R}^m \setminus \{0\})$ such that

$$u\delta(Q) = \tilde{u}\delta(Q).$$

We set

$$S := 2\sqrt{r^2 - Q}\Big(Q\frac{\partial^2}{\partial Q^2} + 2\frac{\partial}{\partial Q}\Big)\big(\tilde{u}\delta(Q)\big).$$

Then, it follows from (2.4.4) and Claim 3.3.3 that

$$\begin{aligned}
&d\hat{\varpi}(\phi(f))(\tilde{u}\delta(Q)) \\
&= \sqrt{-1}\Big(\sqrt{r^2 - Q}\Big(\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{m-1}{2r}\frac{\partial}{\partial r} + \frac{\Delta_{S^{m-1}}}{2r^2}\Big)\tilde{u}\Big)\delta(Q) - S \\
&= \sqrt{-1}\Big(1 + O\big(\big(\frac{Q}{r^2}\big)\big)\Big)\Big(\frac{r}{2}\frac{\partial^2}{\partial r^2} + \frac{m-1}{2}\frac{\partial}{\partial r} + \frac{\Delta_{S^{m-1}}}{2r}\Big)u \cdot \delta(Q) - S \\
&= \sqrt{-1}\Big(\frac{r}{2}\frac{\partial^2}{\partial r^2} + \frac{m-1}{2}\frac{\partial}{\partial r} + \frac{\Delta_{S^{m-1}}}{2r}\Big)u \cdot \delta(Q) - S \\
&= \frac{\sqrt{-1}}{2}r(\Delta_{\mathbb{R}^m}u)\delta(Q) - S.
\end{aligned}$$

At the second last equality, we used the formula $Q\delta(Q) = 0$. In the following claim, we shall show $S = 0$. Now, the proof of (3.3.5) is complete. Hence, we have shown Lemma 3.3.1. $\square$

Claim 3.3.4. *For any $\tilde{u} \in C_0^\infty(\mathbb{R}^{m+1} \setminus \{0\})$, we have*

$$\Big(Q\frac{\partial^2}{\partial Q^2} + 2\frac{\partial}{\partial Q}\Big)\big(\tilde{u}\delta(Q)\big) = 0.$$

Proof of Claim 3.3.4. By the Leibniz rule, the left-hand side amounts to

$$\frac{\partial^2 \tilde{u}}{\partial Q^2}Q\delta(Q) + 2\frac{\partial \tilde{u}}{\partial Q}\Big(Q\frac{\partial}{\partial Q}\delta(Q) + \delta(Q)\Big) + \tilde{u}\Big(Q\frac{\partial^2}{\partial Q^2}\delta(Q) + 2\frac{\partial}{\partial Q}\delta(Q)\Big).$$

Hence we see that this equals 0 in light of the formulas:

$$Q\delta(Q) = 0, \quad Q\frac{\partial}{\partial Q}\delta(Q) = -\delta(Q), \quad Q\frac{\partial^2}{\partial Q^2}\delta(Q) = -2\frac{\partial}{\partial Q}\delta(Q).$$

$\square$

3.4 Central element Z of $\mathfrak{k}_{\mathbb{C}}$

We extend the Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{so}(m+1,2)$ (see (3.3.2)) to the complex Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(m+1,\mathbb{C})$. Consider a generator of $\mathfrak{so}(2,\mathbb{C})$ given by

$$z := \frac{\sqrt{-1}}{2}(e-f) = \frac{\sqrt{-1}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.4.1}$$

We set

$$Z := \phi(z) \in \sqrt{-1}\mathfrak{g}. \tag{3.4.2}$$

In light of (3.3.2) and (2.1.1), we have

$$Z = \frac{\sqrt{-1}}{2}(\overline{N}_{m+1} - N_{m+1}) \tag{3.4.3}$$

$$= \sqrt{-1}(E_{m+1,m+2} - E_{m+2,m+1}). \tag{3.4.4}$$

Hence, $\sqrt{-1}Z$ is contained in the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k} \simeq \mathfrak{so}(m+1) \oplus \mathfrak{so}(2)$. If $m > 1$, then $\mathfrak{c}(\mathfrak{k})$ is of one dimension, and $\sqrt{-1}Z$ generates $\mathfrak{c}(\mathfrak{k})$. By (3.4.1), we have $e^{\sqrt{-1}tZ} = I_{m+3}$ in $SO_0(m+1,2)$ if and only if $t \in 2\pi\mathbb{Z}$. Hence, $e^{\sqrt{-1}tZ} = 1$ in $G = SO_0(m+1,2)^{\sim}$ if and only if $t \in 4\pi\mathbb{Z}$ (see (2.3.1)). Therefore, we have the following lemma:

Lemma 3.4.1. *The Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{so}(m+1,2)$ (see (3.3.1)) lifts to an injective Lie group homomorphism $SL(2,\mathbb{R}) \to G$.*

By the expression (3.4.4) of Z and by the K-isomorphism (3.2.2), we have:

Lemma 3.4.2. *$d\pi_+(Z)$ acts on W_a as a scalar multiplication of $-(a+\frac{m-1}{2})$.*

Combining (3.4.1), (3.4.2), and Lemma 3.3.1, we readily get the following lemma:

Lemma 3.4.3. *The differential operator $d\pi_+(Z)$ takes the form:*

$$d\pi_+(Z) = |x|\Big(\frac{\Delta}{4} - 1\Big).$$

Remark 3.4.4. *$d\pi_+(Z)$ coincides with the operator D in Introduction. In particular, D is a self-adjoint operator on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ because $\sqrt{-1}Z \in \mathfrak{g}$ and π_+ is a unitary representation.*

Remark 3.4.5 (Weil representation). *For the Weil representation of G', the Lie algebra homomorphism $\varphi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sp}(m,\mathbb{R})$ (see Remark 3.3.2) sends z to*

$$Z' := \varphi(z) = \frac{\sqrt{-1}}{2}\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \in \sqrt{-1}\mathfrak{g}'.$$

Hence $\sqrt{-1}Z'$ is a central element of $\mathfrak{k}' \simeq \mathfrak{u}(m)$. The differential operator $d\varpi(Z')$ amounts to the Hermite operator (see [16, §6 (d)])

$$\mathcal{D} = \frac{1}{4}(\Delta - |x|^2).$$

See the table in Subsection 1.4 for the differential operators corresponding to e, f and h.

3.5 Proof of Proposition 3.2.1

This subsection gives a proof of Proposition 3.2.1.

1) Let us show $j_{a,l}(\phi) = f_{a,l}\phi \in L^1(\mathbb{R}^m, \frac{dx}{|x|}) \cap L^2(\mathbb{R}^m, \frac{dx}{|x|})$ for any $\phi \in \mathcal{H}^l(\mathbb{R}^m)$. By the definition (3.2.4) of $f_{a,l}$, $f_{a,l}(r)$ is regular at $r = 0$, and $f_{a,l}(r)$ decays exponentially as r tends to infinity. Therefore, $f_{a,l} \in L^1(\mathbb{R}_+, r^{m-2}dr) \cap L^2(\mathbb{R}_+, r^{m-2}dr)$. Since our measure $\frac{dx}{|x|}$ has the form

$$\frac{dx}{|x|} = r^{m-2} dr d\omega, \tag{3.5.1}$$

with respect to the polar coordinate (3.2.5), we have shown $j_{a,l}(\phi) \in L^1(\mathbb{R}^m, \frac{dx}{|x|}) \cap L^2(\mathbb{R}^m, \frac{dx}{|x|})$.

2) We set $H_{a,l} := j_{a,l}\big(\mathcal{H}^l(\mathbb{R}^m)\big)$. Obviously, $H_{a,l}$ is isomorphic to $\mathcal{H}^l(\mathbb{R}^m)$ as R-modules. To see $W_{a,l} = H_{a,l}$, it is sufficient to show the following inclusion:

$$H_{a,l} \subset W_a, \tag{3.5.2}$$

because $W_{a,l}$ is characterized as the unique subspace of W_a such that $W_{a,l} \simeq \mathcal{H}^l(\mathbb{R}^m)$ as R-modules.

To see (3.5.2), we recall from (3.2.2) that W_a is characterized as the unique subspace of $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ on which $d\pi_+(Z)$ acts with eigenvalue $-(a + \frac{m-1}{2})$. Thus the inclusive relation (3.5.2) will be proved if we show the following lemma:

Lemma 3.5.1. *The operator $d\pi_+(Z)$ acts on $H_{a,l}$ as a scalar multiplication $-(a + \frac{m-1}{2})$. In other words, we have*

$$d\pi_+(Z)(f_{a,l}\phi) = -\Big(a + \frac{m-1}{2}\Big) f_{a,l}\phi. \tag{3.5.3}$$

Proof of Lemma 3.5.1. Writing the differential operator $d\pi_+(Z)$ (see Lemma 3.4.3) in terms of the polar coordinate and using the definition of $f_{a,l}$ (see (3.2.4)), we see that the equation (3.5.3) is equivalent to

$$\left(\frac{r}{4}\frac{\partial^2}{\partial r^2}+\frac{m-1}{4}\frac{\partial}{\partial r}+\frac{\Delta_{S^{m-1}}}{4r}-r+\left(a+\frac{m-1}{2}\right)\right)\left(\psi(4r)r^l e^{-2r}\phi(\omega)\right)=0, \tag{3.5.4}$$

for $\psi(r) := L_{a-l}^{m-2+2l}(r)$. The equation (3.5.4) amounts to

$$\Big(4r\psi''(4r)+(m-1+2l-4r)\psi'(4r)+(a-l)\psi(4r)\Big)r^l e^{-2r}\phi(\omega)=0.$$

This is nothing but Laguerre's differential equation (8.1.2) with $n = a - l, \alpha = m - 2 + 2l$. Now the lemma follows. □

3.6 One parameter holomorphic semigroup $\pi_+(e^{tZ})$

It follows from Lemma 3.4.2 that for $t \in \mathbb{C}$ the operator

$$\pi_+(e^{tZ}) := \exp d\pi_+(tZ) = \sum_{j=0}^{\infty}\frac{1}{j!}d\pi_+(tZ)^j \tag{3.6.1}$$

acts on $W_{a,l}$ as a scalar multiplication of $e^{-(a+\frac{m-1}{2})t}$ for any $0 \le l \le a$, of which the absolute value does not exceed 1 if $\operatorname{Re} t \ge 0$. In light of the direct sum decomposition (3.2.1), if $\operatorname{Re} t \ge 0$ then the linear map $\pi_+(e^{tZ}) : L^2(\mathbb{R}^m, \frac{dx}{|x|})_K \to L^2(\mathbb{R}^m, \frac{dx}{|x|})_K$ extends to a continuous operator (we use the same notation $\pi_+(e^{tZ})$) on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$. Furthermore, it is a contraction operator if $\operatorname{Re} t > 0$. We summarize some of basic properties of $\pi_+(e^{tZ})$:

Proposition 3.6.1. 1) *The map*

$$\{t \in \mathbb{C} : \operatorname{Re} t \ge 0\} \times L^2(\mathbb{R}^m, \frac{dx}{|x|}) \to L^2(\mathbb{R}^m, \frac{dx}{|x|}), \quad (t, f) \mapsto \pi_+(e^{tZ})f \tag{3.6.2}$$

is continuous.

2) *For a fixed t such that $\operatorname{Re} t \ge 0$, $\pi_+(e^{tZ})$ is characterized as the continuous operator from $L^2(\mathbb{R}^m, \frac{dx}{|x|}) \to L^2(\mathbb{R}^m, \frac{dx}{|x|})$ satisfying*

$$\pi_+(e^{tZ})u = e^{-(a+\frac{m-1}{2})t}u, \tag{3.6.3}$$

for any $u \in W_{a,l} = \{f_{a,l}\phi : \phi \in \mathcal{H}^l(\mathbb{R}^m)\}$ (see Proposition 3.2.1) and for any $l, a \in \mathbb{N}$ such that $0 \le l \le a$.

3) *The operator norm* $\|\pi_+(e^{tZ})\|$ *of* $\pi_+(e^{tZ})$ *is* $e^{-\frac{m-1}{2}\operatorname{Re} t}$.
4) *If* $\operatorname{Re} t > 0$, $\pi_+(e^{tZ})$ *is a Hilbert–Schmidt operator.*
5) *If* $t \in \sqrt{-1}\mathbb{R}$, *then* $\pi_+(e^{tZ})$ *is a unitary operator.*

Remark 3.6.2. *We define a subset* $\Gamma_+ := \{tZ : t > 0\}$ *in* $\sqrt{-1}\mathfrak{g}$. *Proposition 3.6.1 indicates how the unitary representation* π_+ *of* G *extends to a holomorphic semigroup on the complex domain* $G \cdot \exp \Gamma_+ \cdot G$ *of* $G_{\mathbb{C}}$. *Our results may be regarded as a part of the Gelfand–Gindikin program, which tries to understand unitary representations of a real semisimple Lie group by means of holomorphic objects on an open subset of* $G_{\mathbb{C}} \setminus G$, *where* $G_{\mathbb{C}}$ *is a complexification of* G *(see [10, 27, 29]).*

Remark 3.6.3 (Weil representation). *In the case of the Weil representation* $(\varpi, L^2(\mathbb{R}^m))$ *of* $G' = Sp(m,\mathbb{R})^\sim$, *the functions* $f'_{a,l}$ *(see Remark 3.2.4) play the same role as* $f_{a,l}$ *because of the following facts:*

1) $\{f'_{a,l} : 0 \le l \le a\}$ *spans a complete orthogonal basis of* $L^2(\mathbb{R}_+, r^{m-1}dr)$ *(cf. Remark 3.2.2).*

2) For any $\phi \in \mathcal{H}^l(\mathbb{R}^m)$, $f'_{a,l}(r)\phi(\omega)$ $(0 \le l \le a)$ *are eigenfunctions of* $d\varpi(Z')$ *with negative eigenvalues* $-(\frac{m}{4} + \frac{l}{2} + \frac{a}{2})$.

Owing to these facts, we obtain a holomorphic semigroup of contraction operators $\varpi(e^{tZ'}) := \exp t(d\varpi(Z'))$ $(\operatorname{Re} t > 0)$ *on* $L^2(\mathbb{R}^m)$. *Since* $d\varpi(Z')$ *coincides with the Hermite operator* $\mathcal{D}$ *(see Remark 3.4.5), this holomorphic semigroup is nothing but the Hermite semigroup (see [16, §5]).*

4 Radial part of the semigroup

This section gives an explicit integral formula for the 'radial part' of the holomorphic semigroup $\pi_+(e^{tZ})$ in the 'Schrödinger model' $L^2(\mathbb{R}^m, |x|^{-1}dx)$. The main result of Section 4 is Theorem 4.1.1. As its applications, we see that the semigroup law $\pi_+(e^{(t_1+t_2)Z}) = \pi_+(e^{t_1 Z}) \circ \pi(e^{t_2 Z})$ gives a simple and representation theoretic proof of the classical Weber's second exponential integral formula on Bessel functions (Corollary 4.5.1), and that taking the boundary value $\lim_{s \downarrow 0} \pi_+(e^{sZ}) = \mathrm{id}$ provides an example of a Dirac sequence (Corollary 4.6.1).

Theorem 4.1.1 will play a key role in Section 5, where we complete the proof of the main theorem of this article, namely, Theorem 5.1.1 that gives an integral formula of the holomorphic semigroup $\pi_+(e^{tZ})$ on $L^2(\mathbb{R}^m, |x|^{-1}dx)$.

4.1 Result of the section

For a complex parameter $t \in \mathbb{C}$ with $\operatorname{Re} t > 0$, we have defined a contraction operator $\pi_+(e^{tZ}) : L^2(\mathbb{R}^m, \frac{dx}{|x|}) \to L^2(\mathbb{R}^m, \frac{dx}{|x|})$ in Proposition 3.6.1.

We recall that Z is a central element in $\mathfrak{k}_{\mathbb{C}}$ (see Subsection 3.4) and that R is a subgroup of K. Therefore, $\pi_+(e^{tZ})$ intertwines with the R-action. On the other hand, the natural action of $R \simeq SO(m)$ gives a direct sum decomposition of the Hilbert space:

$$L^2(\mathbb{R}^m, \frac{dx}{|x|}) \simeq \sum_{l=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}_+, r^{m-2}dr) \otimes \mathcal{H}^l(\mathbb{R}^m). \tag{4.1.1}$$

Hence, by Schur's lemma, there exists a family of continuous operators parametrized by $l \in \mathbb{N}$:

$$\pi_{+,l}(e^{tZ}) : L^2(\mathbb{R}_+, r^{m-2}dr) \to L^2(\mathbb{R}_+, r^{m-2}dr) \tag{4.1.2}$$

such that $\pi_+(e^{tZ})$ is diagonalized according to the direct sum decomposition (4.1.1) as follows:

$$\pi_+(e^{tZ}) = \sum_{l=0}^{\infty}{}^{\oplus} \pi_{+,l}(e^{tZ}) \otimes \mathrm{id}\,. \tag{4.1.3}$$

The goal of this section is to give an explicit integral formula of $\pi_{+,l}(e^{tZ})$ on $L^2(\mathbb{R}_+, r^{m-2}dr)$ for $l \in \mathbb{N}$. We note that $\pi_{+,l}(e^{tZ})$ is a unitary operator if $\operatorname{Re} t = 0$ because so is $\pi_+(e^{tZ})$. Likewise, $\pi_{+,l}(e^{tZ})$ is a Hilbert–Schmidt operator if $\operatorname{Re} t > 0$ because so is $\pi_+(e^{tZ})$.

We now introduce the following subset of $\mathbb{C}$:

$$\Omega := \{t \in \mathbb{C} : \operatorname{Re} t \geq 0\} \setminus 2\pi\sqrt{-1}\mathbb{Z}, \tag{4.1.4}$$

and define a family of analytic functions $K_l^+(r, r'; t)$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times \Omega$ by the formula: for $l = 0, 1, 2, \dots,$

$$\begin{aligned} K_l^+(r, r'; t) &:= \frac{2e^{-2(r+r')\coth\frac{t}{2}}}{\sinh\frac{t}{2}} (rr')^{-\frac{m-2}{2}} I_{m-2+2l}\Big(\frac{4\sqrt{rr'}}{\sinh\frac{t}{2}}\Big) \\ &= \frac{2^{m-1+2l} e^{-2(r+r')\coth\frac{t}{2}} (rr')^l}{(\sinh\frac{t}{2})^{m-1+2l}} \tilde{I}_{m-2+2l}\Big(\frac{4\sqrt{rr'}}{\sinh\frac{t}{2}}\Big). \end{aligned} \tag{4.1.5}$$

Here, $\tilde{I}_\nu(z) := (\frac{z}{2})^{-\nu} I_\nu(z)$, and $I_\nu(z)$ denotes the I-Bessel function (see Subsection 8.5). We note that the denominator $\sinh\frac{t}{2}$ is nonzero for $t \in \Omega$.

We are ready to state the integral formula of $\pi_{+,l}(e^{tZ})$ for $t \in \Omega$:

Theorem 4.1.1 (Radial part of the semigroup). 1) *For* $\operatorname{Re} t > 0$, *the Hilbert–Schmidt operator* $\pi_{+,l}(e^{tZ})$ *on* $L^2(\mathbb{R}_+, r^{m-2}dr)$ *is given by the following integral transform:*

$$(\pi_{+,l}(e^{tZ})f)(r) = \int_0^\infty K_l^+(r, r'; t) f(r') r'^{m-2} dr'. \tag{4.1.6}$$

The right-hand side converges absolutely for $f \in L^2(\mathbb{R}_+, r^{m-2}dr)$.

2) *If* $t \in \sqrt{-1}\mathbb{R}$ *but* $t \notin 2\pi\sqrt{-1}\mathbb{Z}$, *then the integral formula (4.1.6) for the unitary operator* $\pi_{+,l}(e^{tZ})$ *holds in the sense of* L^2*-convergence. Furthermore, the right-hand side converges absolutely if* f *is a finite linear combination of* $f_{a,l}$ $(a = l, l+1, \cdots)$.

Remark 4.1.2. *Let us compare Theorem 4.1.1 with the corresponding result for the Weil representation* ϖ *of* $G' = Sp(m, \mathbb{R})^\sim$ *realized as the Schrödinger model* $L^2(\mathbb{R}^m)$. *According to the direct sum decomposition of the Hilbert space:*

$$L^2(\mathbb{R}^m) \simeq \sum_{l=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}_+, r^{m-1}dr) \otimes \mathcal{H}^l(\mathbb{R}^m),$$

there exists a family of continuous operators $\varpi_l(e^{tZ'})$ $(\operatorname{Re} t > 0)$ *such that the holomorphic semigroup* $\varpi(e^{tZ'})$ *has the following decomposition:*

$$\varpi(e^{tZ'}) = \sum_{l=0}^{\infty}{}^{\oplus} \varpi_l(e^{tZ'}) \otimes \operatorname{id}.$$

Then, by an analogous computation to Theorem 4.1.1, we find the kernel function of the semigroup $\varpi_l(e^{tZ'})$ *is given by*

$$\mathcal{K}_l(r, r'; t) := \frac{e^{-\frac{1}{2}(r^2 + r'^2)\coth\frac{t}{2}}}{\sinh\frac{t}{2}} (rr')^{-\frac{m-2}{2}} I_{\frac{m-2+2l}{2}}\Big(\frac{rr'}{\sinh\frac{t}{2}}\Big).$$

The relation between the kernel function $\mathcal{K}$ *(Mehler kernel) of* $\varpi(e^{tZ'})$ *and* $\mathcal{K}_l(r, r'; t)$ *will be discussed in Remark 5.7.2.*

This section is organized as follows. In Subsection 4.3, we give a proof of Theorem 4.1.1 for the case $\operatorname{Re} t > 0$, which is based on a computation of the kernel function by means of the infinite sum of the eigenfunctions. In Subsection 4.4, by taking the analytic continuation, the case $\operatorname{Re} t = 0$ is proved. Applications of Theorem 4.1.1 to special function theory are discussed in Subsections 4.5 and 4.6.

4.2 Upper estimate of the kernel function

In this subsection, we shall give an upper estimate of the kernel function $K_l^+(r, r'; t)$.

For $t = x + \sqrt{-1}y$, we set

$$\alpha(t) := \frac{\sinh x}{\cosh x - \cos y}, \tag{4.2.1}$$

$$\beta(t) := \frac{\cos \frac{y}{2}}{\cosh \frac{x}{2}}. \tag{4.2.2}$$

Then, an elementary computation shows

$$\operatorname{Re} \coth \frac{t}{2} = \alpha(t), \tag{4.2.3}$$

$$\operatorname{Re} \frac{1}{\sinh \frac{t}{2}} = \alpha(t)\beta(t). \tag{4.2.4}$$

For $t \in \Omega$ (see (4.1.4) for definition), we have $\cosh x - \cos y > 0$, and then,

$$\alpha(t) \geq 0 \quad \text{and} \quad |\beta(t)| < 1. \tag{4.2.5}$$

If $\operatorname{Re} t > 0$, then

$$\alpha(t) > 0. \tag{4.2.6}$$

For later purposes, we prepare:

Lemma 4.2.1. *If $y \in \mathbb{R}$ satisfies $|y| \leq 4\sqrt{rr'}$, then, for $t \in \Omega$, we have the following estimate for some constant C:*

$$\left| e^{-2(r+r') \coth \frac{t}{2}} \widetilde{I}_\nu\left(\frac{y}{\sinh \frac{t}{2}}\right) \right| \leq C e^{-2\alpha(t)(1-|\beta(t)|)(r+r')}. \tag{4.2.7}$$

Proof. Using the upper estimate of the I-Bessel function (see Lemma 8.5.1),

$$|\widetilde{I}_\nu\left(\frac{y}{\sinh \frac{t}{2}}\right)| \leq C e^{|y| |\operatorname{Re} \frac{1}{\sinh \frac{t}{2}}|},$$

we have

$$\begin{aligned} \left| e^{-2(r+r') \coth \frac{t}{2}} \widetilde{I}_\nu\left(\frac{y}{\sinh \frac{t}{2}}\right) \right| &\leq C e^{-2(r+r') \operatorname{Re} \coth \frac{t}{2} + |y| \operatorname{Re} \left| \frac{1}{\sinh \frac{t}{2}} \right|} \\ &\leq C e^{-2(r+r')\alpha(t) + 4\sqrt{rr'}\alpha(t)|\beta(t)|} \\ &\leq C e^{-2\alpha(t)(1-|\beta(t)|)(r+r')}. \end{aligned}$$

Here, the last inequality follows from

$$r + r' - 2|\beta(t)|\sqrt{rr'} \geq (1 - |\beta(t)|)(r + r') \tag{4.2.8}$$

for $t \in \Omega$. Thus Lemma is proved. $\square$

Now we state a main result of this subsection:

Lemma 4.2.2. *Let $l \in \mathbb{N}$ and $m \geq 2$.*
1) *There exists a constant $C > 0$ such that*

$$|K_l^+(r, r'; t)| \leq \frac{C(rr')^l e^{-2\alpha(t)(1-|\beta(t)|)(r+r')}}{|\sinh \frac{t}{2}|^{m-1+2l}}. \tag{4.2.9}$$

for any $r, r' \in \mathbb{R}_+$ and $t \in \Omega$.
2) *If $\operatorname{Re} t > 0$, then $K_l^+(\cdot, \cdot; t) \in L^2((\mathbb{R}_+)^2, (rr')^{m-2} dr dr')$.*
3) *If $\operatorname{Re} t > 0$, then for a fixed $r > 0$, we have $K_l^+(r, \cdot; t) \in L^2(\mathbb{R}_+, r'^{m-2} dr')$.*

Proof. 1) By the definition of K_l^+ (see (4.1.5)), we have

$$|K_l^+(r, r'; t)| = \frac{2^{m-1+2l}(rr')^l}{|\sinh \frac{t}{2}|^{m-1+2l}} \left| e^{-2(r+r')\coth \frac{t}{2}} \widetilde{I}_{m-2+2l}\Big(\frac{4\sqrt{rr'}}{\sinh \frac{t}{2}}\Big) \right|.$$

Now (4.2.9) follows from Lemma 4.2.1 by substituting $\nu = m - 2 + 2l$ and $y = 4\sqrt{rr'}$.

Since $\alpha(t) > 0$ for $\operatorname{Re} t > 0$, the statements 2) and 3) hold by 1). $\square$

4.3 Proof of Theorem 4.1.1 (Case $\operatorname{Re} t > 0$)

We recall from Remark 3.2.2 that

$$f_{a,l}(r) = L_{a-l}^{m-2+2l}(4r) r^l e^{-2r} \quad (a = l, l+1, \ldots)$$

forms a complete orthogonal basis of $L^2(\mathbb{R}_+, r^{m-2} dr)$. Further, by the orthogonal relation of the Laguerre polynomials $L_m^\alpha(x)$ (see (8.1.3)), we have the normalization of $\{f_{a,l}\}$ as follows:

Lemma 4.3.1. *For integers $a, b \geq l$, we have*

$$\int_0^\infty f_{a,l}(r) f_{b,l}(r) r^{m-2} dr = \begin{cases} 0 & \text{if } a \neq b, \\ \frac{\Gamma(m-1+a+l)}{4^{m-1+2l}\Gamma(a-l+1)} & \text{if } a = b. \end{cases} \tag{4.3.1}$$

We rewrite (3.2.1) by using Proposition 3.2.1 as follows:

$$\begin{aligned}
L^2(\mathbb{R}^m, \frac{dx}{|x|})_K &= \bigoplus_{a=0}^{\infty}(\bigoplus_{l=0}^{a} W_{a,l}) \\
&= \bigoplus_{l=0}^{\infty}(\bigoplus_{a=l}^{\infty} W_{a,l}) \\
&= \bigoplus_{l=0}^{\infty}\Big((\bigoplus_{a=l}^{\infty} \mathbb{C} f_{a,l}) \otimes \mathcal{H}^l(\mathbb{R}^m)\Big). \qquad (4.3.2)
\end{aligned}$$

It follows from Proposition 3.6.1 (2) and the definition (4.1.3) of $\pi_{+,l}(e^{tZ})$ that

$$\pi_{+,l}(e^{tZ}) f_{a,l} = e^{-(a+\frac{m-1}{2})t} f_{a,l} \quad (a = l, l+1, \cdots).$$

Therefore, the kernel function $K_l^+(r, r'; t)$ of $\pi_{+,l}(e^{tZ})$ can be written as the infinite sum:

$$K_l^+(r, r'; t) = \sum_{a=l}^{\infty} \frac{e^{-(a+\frac{m-1}{2})t} f_{a,l}(r)\overline{f_{a,l}(r')}}{||f_{a,l}||^2_{L^2(\mathbb{R}_+, r^{m-2}dr)}}. \qquad (4.3.3)$$

Since $\pi_{+,l}(e^{tZ})$ is a Hilbert–Schmidt operator if $\operatorname{Re} t > 0$, the right-hand side converges in $L^2((\mathbb{R}_+)^2, (rr')^{m-2}drdr')$, and therefore converges for almost all $(r, r') \in (\mathbb{R}_+)^2$.

Let us compute the infinite sum (4.3.3). For this, we set

$$\kappa(r, r'; t) := \sum_{a=l}^{\infty} \frac{\Gamma(a-l+1)}{\Gamma(m-1+a+l)} L_{a-l}^{m-2+2l}(4r) L_{a-l}^{m-2+2l}(4r') e^{-(a-l)t}.$$

Then, it follows from (4.3.1) that we have

$$K_l^+(r, r'; t) = 4^{m-1+2l}(rr')^l e^{-2(r+r')} e^{-(l+\frac{m-1}{2})t} \kappa(r, r'; t). \qquad (4.3.4)$$

Now, we apply the Hille–Hardy formula (see (8.1.4)) with $\alpha = m-2+2l, n = a-l, x = 4r, y = 4r'$, and $w = e^{-t}$. We note that $|w| = e^{-\operatorname{Re} t} < 1$ by the assumption $\operatorname{Re} t > 0$. Then we have

$$\begin{aligned}
\kappa(r, r'; t) &= \frac{e^{\frac{(4r+4r')e^{-t}}{1-e^{-t}}}(-16rr'e^{-t})^{-\frac{m-2}{2}+l}}{1-e^{-t}} J_{m-2+2l}\Big(\frac{2\sqrt{-16rr'e^{-t}}}{1-e^{-t}}\Big) \\
&= \frac{(rr')^{-\frac{m-1}{2}+l} e^{-2(r+r')\frac{2e^{-t}}{1-e^{-t}}} e^{(\frac{m-2}{2}+l)t}}{4^{m-2+2l}(1-e^{-t})} I_{m-2+2l}\Big(\frac{4\sqrt{rr'}}{\sinh\frac{t}{2}}\Big).
\end{aligned}$$

Hence, the formula (4.1.5) is proved.

Therefore, the right-hand side of (4.1.6) converges absolutely by the Cauchy–Schwarz inequality because $K_l^+(r,r';t) \in L^2(\mathbb{R}_+, r'^{m-2}dr')$ for any $r > 0$ and $\operatorname{Re} t > 0$ (see Lemma 4.2.2 (3)).

Remark 4.3.2. *The special functions and related formulas that arise in the analysis of the radial part have a scheme of generalization from $SL(2,\mathbb{R})\tilde{}$ to $G = SO_0(m+1,2)\tilde{}$ and $G' = Sp(m,\mathbb{R})\tilde{}$. This scheme is illustrated as follows:*

$SL(2,\mathbb{R})\tilde{}$	$\Rightarrow$	G *or* G'
Segal–Shale–Weil representation	$\Rightarrow$	*minimal representation*
Hermite polynomials	$\Rightarrow$	*Laguerre polynomials*
Mehler's formula	$\Rightarrow$	*Hille–Hardy formula*

See [17, p 116, Exercise 5 (d)] for the $SL(2,\mathbb{R})$ case. Owing to the reduction formula of Laguerre polynomials to Hermite polynomials (see (8.2.1) *and* (8.2.2)*), the radial part $f'_{a,l}(r)$ (see Remark 3.2.4) for the Weil representation of $G' = Sp(m,\mathbb{R})\tilde{}$ collapses to a constant multiple of $H_{2a-l}(r)e^{-\frac{r^2}{2}}$ $(l = 0, 1)$ if $m = 1$.*

Remark 4.3.3. *In [24, Chapter 2], W. Myller-Lebedeff proved the following integral formula:*

$$\int_0^\infty K_l(r,r';t) f_{a,l}(r') r'^{m-2} dr' = e^{-(a+\frac{m-1}{2})t} f_{a,l}(r) \quad \text{for } a \geq l \geq 0. \tag{4.3.5}$$

In view of (4.1.3) *and Proposition 3.6.1 (2), the formula* (4.1.6) *in Theorem 4.1.1 implies* (4.3.5) *and vice versa. The proof of [24] is completely different from ours. Here is a brief sketch: For the partial differential operator*

$$L := \frac{\partial^2}{\partial x^2} + \frac{\alpha+1}{x}\frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial t} \quad \alpha > 0,$$

one has the following identity using Green's formula,

$$\iint_D (vLu - uL^*v)dtdx = \int_{\partial D} (v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} + \frac{\alpha+1}{x}uv)dt + \frac{1}{x}uvdx, \tag{4.3.6}$$

for a domain $D \subset \mathbb{R}^2$. Here, L^* denotes the (formal) adjoint of L. Now, we take $u(x,t) := t^n L_n^\alpha(\frac{x}{t}), v(x,t) := (\frac{x}{\xi})^{\frac{\alpha}{2}} \frac{x}{\tau-t} e^{-\frac{x+\xi}{\tau-t}} I_\alpha(\frac{2\sqrt{x\xi}}{\tau-t}), n \in \mathbb{N}, \tau, \xi \in \mathbb{R}$ as solutions to $Lu = 0, L^*v = 0$ respectively, and the domain D as a rectangular domain $D := \{(x,t) \in \mathbb{R}^2 : 0 < x < \infty, t_1 < t < t_2\}$ for some

$t_1, t_2 \in \mathbb{R}$. Then by the decay properties of u and v, the integrands in the right-hand side of (4.3.6) vanish on $x = 0$ and $x = \infty$. Since the integral of the left-hand side of (4.3.6) vanishes, the integral $\int_0^\infty \frac{1}{x} u(x,t)v(x,t)dx$ becomes constant with respect to t. By taking the limit $t \to \tau$, we have $\lim_{t\to\tau} \int_0^\infty \frac{1}{x} u(x,t)v(x,t)dx = u(\xi,\tau)$ since $v(x,\tau)$ is proved to be a Dirac delta function. Hence we obtain

$$\int_0^\infty \frac{1}{x} u(x,t)v(x,t)dx = u(\xi,\tau),$$

which coincides with (4.3.5) by a suitable change of variables.

4.4 Proof of Theorem 4.1.1 (Case $\operatorname{Re} t = 0$)

Suppose $t \in \sqrt{-1}\mathbb{R}$. Then, $\pi_{+,l}(e^{tZ})$ is a unitary operator on $L^2(\mathbb{R}_+, r^{m-2}dr)$. Suppose furthermore $t \notin 2\pi\sqrt{-1}\mathbb{Z}$. For $\varepsilon > 0$ and $f \in L^2(\mathbb{R}_+, r^{m-2}dr)$, we have from Theorem 4.1.1 (1)

$$\pi_{+,l}(e^{(\varepsilon+t)Z})f = \int_0^\infty K_l^+(r, r'; \varepsilon + t) f(r') r'^{m-2} dr'.$$

By Proposition 3.6.1 (1), the left-hand side converges to $\pi_{+,l}(e^{tZ})f$ in $L^2(\mathbb{R}_+, r^{m-2}dr)$ as ε tends to 0. For the right-hand side, we have:

Claim 4.4.1. *For $t \in \sqrt{-1}\mathbb{R} \setminus 2\pi\sqrt{-1}\mathbb{Z}$,*

$$\lim_{\varepsilon \downarrow 0} \int_0^\infty K_l^+(r, r'; \varepsilon + t) f_{a,l}(r') r'^{m-2} dr' = \int_0^\infty K_l^+(r, r'; t) f_{a,l}(r') r'^{m-2} dr'$$

and the right-hand side converges absolutely.

Proof. If $\varepsilon \in \mathbb{R}$ and $t \in \sqrt{-1}\mathbb{R}$ then

$$\left|\sinh \frac{\varepsilon + t}{2}\right|^2 = \left|\sinh \frac{\varepsilon}{2}\right|^2 + \left|\sinh \frac{t}{2}\right|^2 \geq \left|\sinh \frac{t}{2}\right|^2.$$

Therefore, it follows from (4.2.9) that

$$|K_l^+(r, r'; \varepsilon + t)| \leq \frac{C(rr')^l}{|\sinh \frac{t}{2}|^{m-1+2l}}$$

if $\varepsilon > 0$ and $t \in \sqrt{-1}\mathbb{R} \setminus 2\pi\sqrt{-1}\mathbb{Z}$, because $\varepsilon + t \in \Omega$ implies $\alpha(\varepsilon + t) \geq 0$ and $|\beta(\varepsilon + t)| < 1$. Therefore, we have

$$|K_l^+(r, r'; \varepsilon + t) f_{a,l}(r') r'^{m-2}| \leq \frac{C r^l r'^{l+m-2} e^{-2r'} |L_{a-l}^{m-2+4l}(4r')|}{|\sinh \frac{t}{2}|^{m-1+2l}}.$$

By the Lebesgue convergence theorem, we have proved Claim. $\square$

Since linear combinations of $f_{a,l}$ span a dense subspace of $L^2(\mathbb{R}_+, r^{m-2}dr)$, 2) is proved. $\square$

4.5 Weber's second exponential integral formula

From the semigroup law:

$$\pi_+(e^{(t_1+t_2)Z}) = \pi_+(e^{t_1 Z}) \circ \pi_+(e^{t_2 Z}) \quad (\operatorname{Re} t_1, \operatorname{Re} t_2 > 0), \tag{4.5.1}$$

we get a representation theoretic proof of classical Weber's second exponential integral for Bessel functions (see [33, §13.31 (1)]):

Corollary 4.5.1. (Weber's second exponential integral) *Let ν be a positive integer, and $\rho, \alpha, \beta > 0$. We have the following integral formula*

$$\int_0^\infty e^{-\rho x^2} J_\nu(\alpha x) J_\nu(\beta x) x dx = \frac{1}{2\rho} \exp\Big(-\frac{\alpha^2+\beta^2}{4\rho}\Big) I_\nu\Big(\frac{\alpha\beta}{2\rho}\Big). \tag{4.5.2}$$

Proof. It follows from the semigroup law (4.5.1) that the integral kernels for $\pi_+(e^{(t_1+t_2)Z})$ and $\pi_+(e^{t_1 Z}) \circ \pi_+(e^{t_2 Z})$ must coincide. Then, from Theorem 4.1.1, we have

$$\int_0^\infty K_l^+(r, s; t_1) K_l^+(s, r'; t_2) s^{m-2} ds = K_l^+(r, r'; t_1 + t_2). \tag{4.5.3}$$

In view of (4.1.5), the formula (4.5.2) is obtained by (4.5.3) by the change of variables,

$$x = \sqrt{s}, \quad \alpha = \frac{4e^{\frac{\pi\sqrt{-1}}{2}}\sqrt{r}}{\sinh\frac{t_1}{2}}, \quad \beta = \frac{4e^{\frac{\pi\sqrt{-1}}{2}}\sqrt{r'}}{\sinh\frac{t_2}{2}}, \quad \nu = m - 2 + 2l,$$
$$\rho = 2\Big(\coth\frac{t_1}{2} + \coth\frac{t_2}{2}\Big).$$

$\square$

4.6 Dirac sequence operators

We shall state another corollary to Theorem 4.1.1. Let ν be a positive integer, $x, y \in \mathbb{R}, s \in \mathbb{C}$ such that $\operatorname{Re} s > 0$. For a function f on $\mathbb{R}$, let $\mathcal{T}_s$ be an operator defined by

$$\mathcal{T}_s : f(x) \mapsto \int_0^\infty A(x, y; s) f(y) dy,$$

with the kernel function

$$A(x,y;s) := (xy)^{\frac{1}{2}} \frac{e^{-\frac{1}{2}(x^2+y^2)\coth s}}{\sinh s} I_\nu\Big(\frac{xy}{\sinh s}\Big). \tag{4.6.1}$$

Then we have the following corollary.

Corollary 4.6.1. 1) *The operators* $\{\mathcal{T}_s : \operatorname{Re} s > 0\}$ *form a semigroup of contraction operators on* $L^2(\mathbb{R}_+, dx)$.

2) *(Dirac sequence)* $\lim_{s\to 0} ||\mathcal{T}_s h - h||_{L^2(\mathbb{R}_+,dx)} = 0$ *holds for all* $h \in L^2(\mathbb{R}_+, dx)$.

Remark 4.6.2. *For sufficiently small* s*, the semigroup* $\{\mathcal{T}_s : \operatorname{Re} s > 0\}$ *behaves like the Hermite semigroup (see [16]) whose kernel is given by the following Gaussian (cf.* (1.2.2)*):*

$$\kappa(x,y;s) = \frac{1}{\sqrt{2\pi \sinh s}} e^{-\frac{x^2}{2}\coth s + \frac{xy}{\sinh s} - \frac{y^2}{2}\coth s}$$

because $I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}} e^z$ *for sufficiently large* z *(see [33, §7.23] for the asymptotic behavior of* $I_\nu(z)$*). Note that it is stated in [16, §5.5] that the Hermite semigroup forms a 'Dirac sequence'.*

Proof. Since we assume ν is a positive integer, $\nu = m - 2 + 2l$ for some $m > 3$ and $l \in \mathbb{Z}$. We change the variables $r = \frac{x^2}{4}$, $r' = \frac{y^2}{4}$, $t = 2s$ and define a unitary map

$$\Phi : L^2(\mathbb{R}_+, r^{m-2}dr) \to L^2(\mathbb{R}_+, dx), \quad (\Phi f)(x) := \Big(\frac{x}{2}\Big)^{\frac{2m-3}{2}} f\Big(\frac{x^2}{4}\Big). \tag{4.6.2}$$

Comparing (4.6.1) with (4.1.5), we have

$$\Phi^{-1} \circ \mathcal{T}_s \circ \Phi f = \pi_{+,l}(e^{2sZ}) f \tag{4.6.3}$$

Thus by Theorem 4.1.1 (1), 1) is proved.

2) We take a limit $s \downarrow 0$ of (4.6.3). By Proposition 3.6.1 (1), the right-hand side equals f. Hence by putting $h := \Phi f$, we have $\lim_{s\downarrow 0} \mathcal{T}_s h = h$. $\square$

5 Integral formula for the semigroup

In this section, we shall give an explicit integral formula for the holomorphic semigroup $\exp(td\pi_+(Z)) = \pi_+(e^{tZ})$ on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ for $\operatorname{Re} t > 0$, or more precisely, for $t \in \Omega = \{t \in \mathbb{C} : \operatorname{Re} t \geq 0\} \setminus 2\pi\sqrt{-1}\mathbb{Z}$ (see (4.1.4)). The main result of this section is Theorem 5.1.1. In particular, we give a proof of Theorem A in Introduction.

5.1 Result of the section

Let $\langle x, x' \rangle$ be the standard inner product of $\mathbb{R}^m$, $|x| := \sqrt{\langle x, x \rangle}$ be the norm. We recall the notation from Subsection 1.1:

$$\psi(x, x') := 2\sqrt{2(|x||x'| + \langle x, x' \rangle)} = 4|x|^{\frac{1}{2}}|x'|^{\frac{1}{2}} \cos \frac{\theta}{2}, \tag{5.1.1}$$

where $\theta \equiv \theta(x, x')$ is the angle between x and x' in $\mathbb{R}^m$. Let us define a kernel function $K^+(x, x'; t)$ on $\mathbb{R}^m \times \mathbb{R}^m \times \Omega$ by the following formula as in Introduction:

$$\begin{aligned} K^+(x, x'; t) &:= \frac{2^{\frac{m-1}{2}} e^{-2(|x|+|x'|) \coth \frac{t}{2}}}{\pi^{\frac{m-1}{2}} \sinh^{\frac{m+1}{2}} \frac{t}{2}} \psi(x, x')^{-\frac{m-3}{2}} I_{\frac{m-3}{2}} \Big(\frac{\psi(x, x')}{\sinh \frac{t}{2}} \Big) \\ &= \frac{2e^{-2(|x|+|x'|) \coth \frac{t}{2}}}{\pi^{\frac{m-1}{2}} \sinh^{m-1} \frac{t}{2}} \tilde{I}_{\frac{m-3}{2}} \Big(\frac{\psi(x, x')}{\sinh \frac{t}{2}} \Big), \end{aligned} \tag{5.1.2}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind and $\tilde{I}_\nu(z) := (\frac{z}{2})^{-\nu} I_\nu(z)$ is an entire function (see Subsection 8.5). We note that $\sinh \frac{t}{2}$ in the denominator is non-zero because $t \notin 2\pi\sqrt{-1}\mathbb{Z}$. Therefore, $K^+(x, x'; t)$ is a continuous function on $\mathbb{R}^m \times \mathbb{R}^m \times \Omega$.

We recall from Proposition 3.6.1 (3) that $\pi_+(e^{tZ})$ is a contraction operator with operator norm $\|\pi_+(e^{tZ})\| = e^{-\frac{m-1}{2} \operatorname{Re} t}$. Here is an integral formula of the holomorphic semigroup $\pi_+(e^{tZ})$:

Theorem 5.1.1 (Integral formula for the semigroup). 1) *For* $\operatorname{Re} t > 0$, *$\pi_+(e^{tZ})$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, and is given by the following integral transform:*

$$(\pi_+(e^{tZ})u)(x) = \int_{\mathbb{R}^m} K^+(x, x'; t) u(x') \frac{dx}{|x|} \quad \text{for } u \in L^2(\mathbb{R}^m, \frac{dx}{|x|}). \tag{5.1.3}$$

Here, the right-hand side converges absolutely.

2) *For $t \in \sqrt{-1}\mathbb{R}$, $\pi_+(e^{tZ})$ is a unitary operator on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$. If $t \in \sqrt{-1}\mathbb{R}$ but $t \notin 2\pi\sqrt{-1}\mathbb{Z}$, then the right-hand side of* (5.1.3) *converges absolutely for any $u \in L^1(\mathbb{R}^m, \frac{dx}{|x|}) \cap L^2(\mathbb{R}^m, \frac{dx}{|x|})$, in particular, for any K-finite vectors in $L^2(\mathbb{R}^m, \frac{dx}{|x|})$. Since K-finite vectors span a dense subspace of $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, the integral formula* (5.1.3) *holds in the sense of L^2-convergence.*

Remark 5.1.2 (Realization on the cone C_+). *Via the isomorphism $L^2(C_+) \simeq L^2(\mathbb{R}^m, \frac{dx}{|x|})$ (see Subsection 3.1), the above formula for $L^2(\mathbb{R}^m, \frac{dx}{|x|})$*

can be readily transferred to the formula of the holomorphic extension $\pi_+(e^{tZ})$ *on* $L^2(C_+)$*. For this, we define a continuous function* $\widetilde{K^+}(\zeta,\zeta';t)$ *on* $C_+ \times C_+ \times \Omega$ *by the following formula:*

$$\widetilde{K^+}(\zeta,\zeta';t) := \frac{2e^{-\sqrt{2}(|\zeta|+|\zeta'|)\coth\frac{t}{2}}}{\pi^{\frac{m-1}{2}}\sinh^{m-1}\frac{t}{2}}\tilde{I}_{\frac{m-3}{2}}\Big(\frac{2\sqrt{2\langle\zeta,\zeta'\rangle}}{\sinh\frac{t}{2}}\Big), \tag{5.1.4}$$

where $|\zeta| := \sqrt{\langle\zeta,\zeta\rangle} = (\zeta_1^2+\cdots+\zeta_{m+1}^2)^{\frac{1}{2}}$*. Then,*

$$(\pi_+(e^{tZ})u)(\zeta) = \int_{C_+}\widetilde{K^+}(\zeta,\zeta';t)u(\zeta')d\mu(\zeta') \quad \textit{for } u \in L^2(C_+). \tag{5.1.5}$$

Remark 5.1.3 (Weil representation). *For the Weil representation* ϖ *of* G'*, the corresponding semigroup of contraction operators is the Hermite semigroup* $\{\varpi(e^{tZ'}) : \operatorname{Re} t > 0\}$ *(see Remark 3.6.3), whose kernel function is given by the Mehler kernel* $\mathcal{K}(x,x';t)$ *(see Fact C in Subsection 1.2; see also [16, §5]).*

The rest of this section is devoted to the proof of Theorem 5.1.1. Let us mention briefly a naive idea of the proof. We observe that the action of $K(= SO(m+1)\times \widetilde{SO(2)})$ on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ is hard to describe because K does not act on $\mathbb{R}^m$. However, the action of its subgroup $R = SO(m)$ has a simple feature, that is, we have the following direct sum decomposition:

$$L^2(\mathbb{R}^m, \frac{dx}{|x|}) \simeq \sum_{l=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}_+, r^{m-2}dr)\otimes \mathcal{H}^l(\mathbb{R}^m). \tag{5.1.6}$$

We have already proved in Theorem 4.1.1 that $K_l^+(r,r';t)$ is the kernel of $\pi_{+,l}(e^{tZ})$ which is the restriction of $\pi_+(e^{tZ})$ in each l-component of the right-hand side of (5.1.6). Theorem 5.1.1 will be proved if we decompose K^+ into K_l^+. This will be carried out in Lemma 5.6.1. An expansion formula of K^+ by K_l^+ is not used in the proof of Theorem 5.1.1, but might be of interest of its own. We shall give it in Subsection 5.7.

5.2 Upper estimates of the kernel function

In this subsection, we give an upper estimate of the kernel function $K^+(x,x';t)$. That parallels Lemma 4.2.2.

Lemma 5.2.1. *Let* $m \geq 2$*. 1) There exists a constant* $C > 0$ *such that*

$$\big|K^+(r\omega, r'\omega';t)\big| \leq \frac{C}{|\sinh\frac{t}{2}|^{m-1}}\, e^{-2\alpha(t)(1-|\beta(t)|)(r+r')}, \tag{5.2.1}$$

for any $r, r' \in \mathbb{R}_+$*,* $\omega, \omega' \in S^{m-1}$*, and* $t \in \Omega$*. Here,* $\alpha(t), \beta(t)$ *are defined in* (4.2.1) *and* (4.2.2)*, respectively.*

2) *If* $\operatorname{Re} t > 0$*, then*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| K^+(x, x'; t) \right|^2 \frac{dx}{|x|} \frac{dx'}{|x'|} < \infty.$$

3) *If* $\operatorname{Re} t > 0$*, then for a fixed* $x \in \mathbb{R}^m$*, we have* $K^+(x, \cdot\,; t) \in L^2(\mathbb{R}^m, \frac{dx'}{|x'|})$*.*

Proof. By the definition (5.1.2) of $K^+(x, x'; t)$, we have

$$|K^+(r\omega, r'\omega'; t)| = \frac{2e^{-2(r+r')\coth\frac{t}{2}}}{\pi^{\frac{m-1}{2}} |\sinh\frac{t}{2}|^{m-1}} \left| \tilde{I}_{\frac{m-3}{2}} \left(\frac{\psi(r\omega, r'\omega')}{\sinh\frac{t}{2}} \right) \right|.$$

By (5.1.1), we have $|\psi(r\omega, r'\omega')| \leq 4\sqrt{rr'}$. Applying Lemma 4.2.1 with $\nu = \frac{m-3}{2}$, we have

$$|K^+(r\omega, r'\omega'; t)| \leq \frac{2Ce^{-\alpha(t)(1-|\beta(t)|)(r+r')}}{\pi^{\frac{m-1}{2}} |\sinh\frac{t}{2}|^{m-1}}.$$

Replacing C with a new constant, we get (5.2.1).

The second and third statements follow from (5.2.1) because $\alpha(t) > 0$ and $|\beta(t)| < 1$ if $\operatorname{Re} t > 0$. □

5.3 Proof of Theorem 5.1.1 (Case $\operatorname{Re} t > 0$)

Suppose $\operatorname{Re} t > 0$. We set

$$(S_t u)(x) := \int_{\mathbb{R}^m} K^+(x, x'; t) u(x') \frac{dx'}{|x'|}. \tag{5.3.1}$$

By Lemma 5.2.1 (2), we observe that S_t is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$, and the right-hand side of (5.3.1) converges absolutely for $u \in L^2(\mathbb{R}^m, \frac{dx}{|x|})$ by the Cauchy–Schwarz inequality and by Lemma 5.2.1 (3).

The remaining assertion of Theorem 5.1.1 (1) is the equality $\pi_+(e^{tZ}) = S_t$. To see this, we observe from the definition (5.1.2) of $K^+(x, x'; t)$ that

$$K^+(kx, kx'; t) = K^+(x, x'; t) \quad \text{for all } k \in R.$$

Therefore, the operator S_t intertwines the R-action, and preserves each summand of (4.1.1). In light of the decomposition (4.1.3) of the operator $\pi_+(e^{tZ})$, the equality $\pi_+(e^{tZ}) = S_t$ will follow from:

Lemma 5.3.1. *Let* $\operatorname{Re} t > 0$. *For every* $l \in \mathbb{N}$, *we have*

$$\pi_{+,l}(e^{tZ}) \otimes \mathrm{id} = S_t|_{L^2(\mathbb{R}_+, r^{m-2}dr)\otimes \mathcal{H}^l(\mathbb{R}^m)}. \tag{5.3.2}$$

We postpone the proof of Lemma 5.3.1 until Subsection 5.6.

Thus, the proof of Theorem 5.1.1 (1) is completed by admitting Lemma 5.3.1.

5.4 Proof of Theorem 5.1.1 (Case $\operatorname{Re} t = 0$)

Suppose $\operatorname{Re} t = 0$. Then, by Lemma 5.2.1 (1), we have

$$\left|K^+(x, x'; t)\right| \le \frac{C}{|\sinh \frac{t}{2}|^{m-1}}$$

because $\alpha(t) = 0$. Therefore, the right-hand side of (5.1.3) converges absolutely for any $u \in L^1(\mathbb{R}^m, \frac{dx}{|x|}) \cap L^2(\mathbb{R}^m, \frac{dx}{|x|})$, as is seen by

$$\int_{\mathbb{R}^m} \left|K^+(x, x'; t) u(x')\right| \frac{dx'}{|x'|} \le \frac{C}{|\sinh \frac{t}{2}|^{m-1}} \int_{\mathbb{R}^m} |u(x')| \frac{dx'}{|x'|} < \infty.$$

By Proposition 3.2.1 (1), we have $L^2(\mathbb{R}^m, \frac{dx}{|x|})_K \subset L^1(\mathbb{R}^m, \frac{dx}{|x|})$. Hence, the right-hand side of (5.1.3) converges absolutely, in particular, for K-finite functions.

Finally, let us show the last statement of (2). Since $W_{a,l} (0 \le l \le a)$ spans $L^2(\mathbb{R}^m, \frac{dx}{|x|})_K$ (see (3.2.1)), it is sufficient to prove

$$\pi_+(e^{tZ}) = S_t \quad \text{on} \quad W_{a,l} \ (0 \le l \le a). \tag{5.4.1}$$

We recall from Proposition 3.2.1 that every vector $u \in W_{a,l}$ is of the form

$$u(r\omega) = f_{a,l}(r)\phi(\omega) \tag{5.4.2}$$

for some $\phi \in \mathcal{H}^l(\mathbb{R}^m)$(see (3.2.4) for the definition). Suppose $\varepsilon > 0$ and $t \in \sqrt{-1}\mathbb{R} \setminus 2\pi\sqrt{-1}\mathbb{Z}$. As in the proof of Claim 4.4.1, we have

$$|K^+(x, x'; \varepsilon + t)| \le \frac{C'}{|\sinh \frac{t}{2}|^{m-1}}$$

and therefore

$$|K^+(x, x'; \varepsilon + t) f_{a,l}(r')\phi(\omega) r'^{m-2}| \le \frac{C r'^{m-2+l} e^{-2r'} |L_{a-l}^{m-2+2l}(4r')|}{|\sinh \frac{t}{2}|^{m-1}}.$$

Here, $C := C' \max_{\omega\in S^{m-1}} |\phi(\omega)|$. Hence, by the dominated convergence theorem, we have

$$\lim_{\varepsilon\downarrow 0} S_{\varepsilon+t}u = S_t u$$

for any $u \in W_{a,l}$.

On the other hand, by Proposition 3.6.1 (1), we have

$$\lim_{\varepsilon\downarrow 0} \pi_+(e^{(\varepsilon+t)Z})u = \pi_+(e^{tZ})u.$$

Since $\pi_+(e^{(\varepsilon+t)Z}) = S_{\varepsilon+t}$ for $\varepsilon > 0$, we have now proved (5.4.1).

5.5 Spectra of an $O(m)$-invariant operator

The rest of this section is devoted to the proof of Lemma 5.3.1.

The orthogonal group $O(m)$ acts on $L^2(S^{m-1})$ as a unitary representation, and decomposes it into irreducible representations as follows:

$$L^2(S^{m-1}) \simeq \sum_{l=0}^{\infty}{}^{\oplus} \mathcal{H}^l(\mathbb{R}^m).$$

Since this is a multiplicity-free decomposition, any $O(m)$-invariant operator S on $L^2(S^{m-1})$ acts on each irreducible component as a scalar multiplication.

The next lemma gives an explicit formula of the spectrum for an $O(m)$-invariant integral operator on $C(S^{m-1})$ in the general setting. This should be known to experts, but for the convenience of the readers, we present it in the following form:

Lemma 5.5.1. *For a continuous function h on the closed interval $[-1,1]$, we consider the following integral transform:*

$$S_h : L^2(S^{m-1}) \to L^2(S^{m-1}), \quad \phi(\omega) \mapsto \int_{S^{m-1}} h(\langle \omega, \omega' \rangle)\phi(\omega')d\omega'. \tag{5.5.1}$$

Then, S_h acts on $\mathcal{H}^l(\mathbb{R}^m)$ by a scalar multiplication of $c_{l,m}(h) \in \mathbb{C}$. The constant $c_{l,m}(h)$ is given by

$$c_{l,m}(h) = \frac{2^{m-2}\pi^{\frac{m-2}{2}} l!}{\Gamma(m-2+l)} \int_0^{\pi} h(\cos\theta)\widetilde{C}_l^{\frac{m-2}{2}}(\cos\theta)\sin^{m-2}\theta d\theta, \tag{5.5.2}$$

where $\widetilde{C}_l^{\frac{m-2}{2}}(x)$ denotes the normalized Gegenbauer polynomial (see (8.3.2)).

Example 5.5.2 (see [15, Introduction, Lemma 3.6]). *For $h(x) := e^{\sqrt{-1}\lambda x}$, $c_{l,m}(h)$ amounts to*

$$c_{l,m}(h) = (2\pi)^{\frac{m}{2}} e^{\frac{\sqrt{-1}}{2}\pi l} \lambda^{-\frac{m-2}{2}} J_{\frac{m-2}{2}+l}(\lambda).$$

Example 5.5.3. *We set $\tilde{I}_\nu(z) = \left(\frac{z}{2}\right)^{-\nu} I_\nu(z)$ (see (8.5.6)). For $h(s) := \tilde{I}_{\frac{m-3}{2}}(\alpha\sqrt{1+s})$, we have*

$$c_{l,m}(h) = 2^{\frac{3m-4}{2}} \pi^{\frac{m-1}{2}} \alpha^{-m+2} I_{m-2+2l}(\sqrt{2}\alpha) \tag{5.5.3}$$

Proof of Example 5.5.3. We apply Lemma 8.5.2 with $\nu = \frac{m-3}{2}$. Then, we have

$$\begin{aligned}&\int_0^\pi \tilde{I}_{\frac{m-3}{2}}(\alpha\sqrt{1+\cos\theta}) \widetilde{C}_l^{\frac{m-2}{2}}(\cos\theta) \sin^{m-2}\theta \, d\theta \\ &= \frac{2^{\frac{m}{2}}\sqrt{\pi}\,\Gamma(m-2+l)}{\alpha^{m-2} l!} I_{m-2+2l}(\sqrt{2}\alpha).\end{aligned}$$

Hence, (5.5.3) follows from Lemma 5.5.1. □

Proof of Lemma 5.5.1. 1) The operator S_h intertwines the $O(m)$-action because $h(\langle k\omega, k\omega'\rangle) = h(\langle \omega, \omega'\rangle)$ for $k \in O(m)$. Hence it follows from Schur's lemma that S_h acts on each irreducible $O(m)$-subspace $\mathcal{H}^l(\mathbb{R}^m)$ by the multiplication of a constant, which we shall denote by $c_{l,m}(h)$ for $l = 0, 1, 2, \ldots$. Thus, we have

$$(S_h\phi)(\omega) = c_{l,m}(h)\phi(\omega) \quad \text{for } \phi \in \mathcal{H}^l(\mathbb{R}^m). \tag{5.5.4}$$

To compute the constant $c_{l,m}(h)$, we use the following coordinate:

$$[0, \pi) \times S^{m-2} \to S^{m-1}, \quad (\theta, \eta) \mapsto \omega = (\cos\theta, \sin\theta \cdot \eta).$$

With this coordinate, we have $d\omega = \sin^{m-2}\theta d\theta d\eta$.

We set $\omega_0 = (1, 0, \cdots, 0)$. Now, we take $\phi(\omega) := \widetilde{C}_l^{\frac{m-2}{2}}(\langle \omega, \omega_0\rangle) \in \mathcal{H}^l(\mathbb{R}^m)$, which is an $O(m-1)$-invariant spherical harmonics. Then the equation (5.5.4) for $\omega = \omega_0$ amounts to

$$\frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})} \int_0^\pi h(\cos\theta) \widetilde{C}_l^{\frac{m-2}{2}}(\cos\theta) \sin^{m-2}\theta d\theta = \frac{\sqrt{\pi}\,\Gamma(m-2+l)}{2^{m-3} l!\,\Gamma(\frac{m-1}{2})} c_{l,m}(h),$$

because $\mathrm{vol}(S^{m-2}) = \frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}$ and $\phi(\omega_0) = \widetilde{C}_l^{\frac{m-2}{2}}(1) = \frac{\sqrt{\pi}\,\Gamma(m-2+l)}{2^{m-3} l!\,\Gamma(\frac{m-1}{2})}$ (see (8.3.3)). Hence, (5.5.2) is proved. □

5.6 Proof of Lemma 5.3.1

This subsection gives a proof of Lemma 5.3.1.

We recall from Theorem 4.1.1 (1) that the kernel function of $\pi_{+,l}(e^{tZ})$ is given by $K_l^+(r, r'; t)$ (see (4.1.5) for definition). Therefore, the equation (5.3.2) is equivalent to the following equation between kernel functions:

Lemma 5.6.1. *For $\phi \in \mathcal{H}^l(\mathbb{R}^m)$, we have*

$$\frac{1}{2}\int_{S^{m-1}} K^+(r\omega, r'\omega'; t)\phi(\omega')d\omega' = K_l^+(r, r'; t)\phi(\omega). \tag{5.6.1}$$

Proof of Lemma 5.6.1. We set

$$h(r, r', t, s) := \frac{2e^{-2(r+r')\coth\frac{t}{2}}}{\pi^{\frac{m-1}{2}}\sinh^{m-1}\frac{t}{2}}\tilde{I}_{\frac{m-3}{2}}\left(\frac{2\sqrt{2rr'(1+s)}}{\sinh\frac{t}{2}}\right). \tag{5.6.2}$$

By the definition (5.1.2) of $K^+(x, x'; t)$, we have

$$K^+(r\omega, r'\omega'; t) = h(r, r', t, \langle\omega, \omega'\rangle).$$

Consider now the integral transform $S_{h(r,r',t,\cdot)}$ on $L^2(S^{n-1})$ with kernel $h(r, r', t, \langle\omega, \omega'\rangle)$. Then it follows from Example 5.5.3 with $\alpha = \frac{2\sqrt{2rr'}}{\sinh\frac{t}{2}}$ that

$$c_{l,m}(h(r, r', t, \cdot)) = 2K_l^+(r, r'; t). \tag{5.6.3}$$

Then, (5.6.1) is a direct consequence of Lemma 5.5.1. $\square$

5.7 Expansion formulas

We recall that the kernel functions for the semigroups $\pi_+(e^{tZ})$ and $\pi_{+,l}(e^{tZ})$ are given by $K^+(r\omega, r'\omega'; t)$ and $K_l^+(r, r'; t)$. In this subsection, we shall give expansion formulas for $K^+(r\omega, r'\omega'; t)$ arising from the decomposition (see (4.1.3))

$$\pi_+(e^{tZ}) = \sum_{l=0}^{\infty}{}^{\oplus}\,\pi_{+,l}(e^{tZ}) \otimes \mathrm{id}\,.$$

Proposition 5.7.1 (Expansion formulas). *Let $m > 1$.*

1) The kernel function $K^+(x, x'; t)$ (see (5.1.2)) has the following expansion:

$$K^+(r\omega, r'\omega'; t) = \frac{1}{\pi^{\frac{m}{2}}}\sum_{l=0}^{\infty}(\frac{m-2}{2}+l)K_l^+(r, r'; t)\widetilde{C}_l^{\frac{m-2}{2}}(\langle\omega, \omega'\rangle). \tag{5.7.1}$$

2) The special value $t = \pi\sqrt{-1}$ for (5.7.1) *yields the expansion formula for the Bessel function:*

$$\tilde{J}_{\nu-\frac{1}{2}}(\sqrt{z}\cos\frac{\theta}{2}) = \frac{2^{4\nu}}{\sqrt{\pi}}\sum_{l=0}^{\infty}(\nu+l)(-1)^l\frac{J_{2\nu+2l}(\sqrt{z})}{z^{\nu}}\widetilde{C}_l^{\nu}(\cos\theta) \tag{5.7.2}$$

for $z \in \mathbb{R}_+, \nu \in \frac{1}{2}\mathbb{Z}$.

Remark 5.7.2 (Weil representation, Gegenbauer's expansion). *Let us compare the above result with the case of the Weil representation of G'. Then, by a similar argument to the proof of Proposition 5.7.1, we can show that the Mehler kernel $\mathcal{K}$ (see Subsection 1.2) has the following decomposition:*

$$\mathcal{K}(r\omega, r'\omega'; t) = \frac{1}{2\pi^{\frac{m}{2}}}\sum_{l=0}^{\infty}(\frac{m-2}{2}+l)\mathcal{K}_l(r,r';t)\widetilde{C}_l^{\frac{m-2}{2}}(\langle\omega,\omega'\rangle) \tag{5.7.3}$$

if $m > 1$.

In light of the formula (see Subsection 1.2)

$$\mathcal{K}(r\omega, r'\omega'; \pi\sqrt{-1}) = \frac{1}{(2\pi\sqrt{-1})^{\frac{m}{2}}}e^{-\sqrt{-1}rr'\langle\omega,\omega'\rangle},$$

the special value at $t = \pi\sqrt{-1}$ for (5.7.3) *yields the following expansion formula for the exponential function known as Gegenbauer's expansion ([9], see also [33, Chapter XI, §11.5]):*

$$e^{\sqrt{-1}z\cos\phi} = 2^{\nu}\sum_{m=0}^{\infty}(\nu+m)\sqrt{-1}^m\frac{J_{\nu+m}(z)}{z^{\nu}}\widetilde{C}_m^{\nu}(\cos\phi) \tag{5.7.4}$$

for $z \in \mathbb{R}, \nu \in \frac{1}{2}\mathbb{Z}$. This formula corresponds to (5.7.2)*.*

If $m = 1$, we have

$$\mathcal{K}(x,x';t) = \frac{1}{2}\big(\mathcal{K}_0(|x|,|x'|;t)\operatorname{id}(x) + \mathcal{K}_1(|x|,|x'|;t)\operatorname{sgn}(x)\big), \tag{5.7.5}$$

which is immediately verified by the formulas for $\mathcal{K}_0, \mathcal{K}_1$ and $\mathcal{K}$ (see Remark 4.1.2).

Proof of Proposition 5.7.1. 1) First we prepare a lemma.

Lemma 5.7.3. *Let $f \in C[-1,1]$. Then f is expanded into the infinite series:*

$$f(x) = \frac{1}{2\pi^{\frac{m}{2}}}\sum_{l=0}^{\infty}\big(\frac{m-2}{2}+l\big)c_{l,m}(f)\widetilde{C}_l^{\frac{m-2}{2}}(x), \tag{5.7.6}$$

where $c_{l,m}(f)$ is the constant defined in (5.5.2)*.*

Proof of Lemma 5.7.3. Applying the expansion formula $f(x) = \sum_{l=0}^{\infty} \alpha_l^{\nu}(f)\widetilde{C}_l^{\nu}(x)$ (see (8.3.5)) with $\nu = \frac{m-2}{2}$, we have

$$\alpha_l^{\frac{m-2}{2}}(f) = \frac{(\frac{m-2}{2}+l)}{2\pi^{\frac{m}{2}}} c_{l,m}(f), \tag{5.7.7}$$

from (5.5.2) and (8.3.6). Hence, we get the lemma. □

Let $h(r, r', t, s)$ be as in (5.6.2). We recall from (5.6.3)

$$c_{l,m}(h(r, r', t, \cdot)) = 2K_l^{+}(r, r'; t).$$

Therefore, by Lemma 5.7.3, we get (5.7.1).

2) Substitute $t = \pi\sqrt{-1}$ and put

$$z := 16rr', \quad \cos\theta := \langle \omega, \omega' \rangle, \quad \nu = \frac{m-2}{2},$$

we get (5.7.2). □

6 The unitary inversion operator

6.1 Result of the section

We define the 'inversion' element $w_0 \in G$ by

$$w_0 := e^{\pi\sqrt{-1}Z}.$$

Then, clearly, w_0 has the following properties:

1) w_0 is of order four.

2) w_0 normalizes $M^{\max}A$ and $\mathrm{Ad}(w_0)\mathfrak{n}^{\max} = \overline{\mathfrak{n}^{\max}}$.

3) The group G is generated by $\overline{P^{\max}}$ and w_0.

We note that if m is odd then $e^{\pi\sqrt{-1}Z}$ is equal to $\begin{pmatrix} I_{m+1} & 0 \\ 0 & -I_2 \end{pmatrix}$ in $SO_0(m+1, 2) = G/\{1, \eta\}$.

This section gives an explicit integral formula of the unitary operator $\pi_+(w_0)$ on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$. In light of $w_0 = e^{\pi\sqrt{-1}Z}$ and $\pi\sqrt{-1} \in \Omega$, we define the following kernel functions by substituting $t = \pi\sqrt{-1}$ into (5.1.2) and (4.1.5), respectively:

$$\begin{aligned} K^{+}(x, x') &:= K^{+}(x, x'; \pi\sqrt{-1}) \\ &= \frac{2}{e^{\frac{m-1}{2}\pi\sqrt{-1}}\pi^{\frac{m-1}{2}}} \sqrt{2\langle \zeta, \zeta' \rangle}^{-\frac{m-3}{2}} J_{\frac{m-3}{2}}(2\sqrt{2\langle \zeta, \zeta' \rangle}), \end{aligned} \tag{6.1.1}$$

$$\begin{aligned} K_l^{+}(r, r') &:= K_l^{+}(r, r'; \pi\sqrt{-1}) \\ &= 2(-1)^l e^{-\frac{m-1}{2}\pi\sqrt{-1}} (rr')^{-\frac{m-2}{2}} J_{m-2+2l}(4\sqrt{rr'}). \end{aligned} \tag{6.1.2}$$

Then, the following result is a direct consequence of Theorems 5.1.1 and 4.1.1.

Theorem 6.1.1 (Integral formula for the unitary inversion operator). *The unitary operator* $\pi_+(w_0) : L^2(\mathbb{R}^m, \frac{dx}{|x|}) \to L^2(\mathbb{R}^m, \frac{dx}{|x|})$ *is given by the following integral transform:*

$$(Tu)(x) := \int_{\mathbb{R}^m} K^+(x, x')u(x')\frac{dx}{|x|}, \quad u \in L^2(\mathbb{R}^m, \frac{dx}{|x|}). \tag{6.1.3}$$

Here, the integral (6.1.3) *converges absolutely for any* $u \in L^1(\mathbb{R}^m, \frac{dx}{|x|}) \cap L^2(\mathbb{R}^m, \frac{dx}{|x|})$*, in particular, for any* K*-finite vector, hence the equation* (6.1.3) *holds in the sense of* L^2*-convergence.*

The substitution of $t = \pi\sqrt{-1}$ into (4.1.3) gives the decomposition

$$\pi_+(w_0) = \sum_{l=0}^{\infty}{}^{\oplus} \pi_{+,l}(w_0) \otimes \mathrm{id}\,.$$

As $\pi_+(w_0)$ is given by the kernel $K^+(x, x')$, so is $\pi_{+,l}(w_0)$ by $K_l^+(r, r')$. Thus, we have:

Theorem 6.1.2 (Radial part of the unitary inversion operator). *The unitary operator* $\pi_{+,l}(w_0) : L^2(\mathbb{R}_+, r^{m-2}dr) \to L^2(\mathbb{R}_+, r^{m-2}dr)$ *is given by the following integral transform:*

$$(T_l f)(r) := \int_0^\infty K_l^+(r, r')f(r')r'^{m-2}dr'. \tag{6.1.4}$$

Here, the integral (6.1.4) *converges absolutely for any* $f \in L^1(\mathbb{R}_+, r^{m-2}dr) \cap L^2(\mathbb{R}_+, r^{m-2}dr)$*. In particular, the equation* (6.1.4) *holds in the sense of* L^2*-convergence.*

Remark 6.1.3 (Weil representation). *In the case of the Weil representation* ϖ *of* G'*, the counterparts of Theorem 6.1.1 and Theorem 6.1.2 can be stated as follows: let* $\omega_0 := e^{\pi\sqrt{-1}Z'}$*.*

We define kernel functions $\mathcal{K}$ *and* $\mathcal{K}_l$ *by the formulas*

$$\mathcal{K}(x, x') := \mathcal{K}(x, x'; \pi\sqrt{-1}) = \frac{1}{(2\pi\sqrt{-1})^{\frac{m}{2}}} e^{-\sqrt{-1}\langle x, x'\rangle},$$

$$\mathcal{K}_l(r, r') := \mathcal{K}_l(r, r'; \pi\sqrt{-1}) = \sqrt{-1}^{-\frac{m-1+2l}{2}} (rr')^{-\frac{m-2}{2}} J_{\frac{m-2+2l}{2}}(rr').$$

Then,

1) The unitary operator $\varpi(w_0) : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ is given by

$$(\varpi(w_0)u)(x) = \int_{\mathbb{R}^m} \mathcal{K}(x, x')u(x')dx'.$$

Hence we see that $\varpi(w_0)$ is nothing but the Fourier transform.

2) The 'radial' part of $\varpi(w_0)$, namely, the unitary operator $\varpi_l(w_0)$: $L^2(\mathbb{R}_+, r^{m-1}dr) \to L^2(\mathbb{R}_+, r^{m-1}dr)$ (see Remark 4.1.2) is given by

$$(\varpi_l(w_0)f)(r) = \int_0^\infty \mathcal{K}_l(r, r')f(r')r'^{m-1}dr'.$$

6.2 Inversion and Plancherel formula

It follows from (3.6.3) that $\pi(w_0)^2 = \pi_+(e^{2\pi\sqrt{-1}Z}) = (-1)^{m+1}\,\mathrm{id}$. Thus, as an immediate consequence of Theorem 6.1.1, we have:

Corollary 6.2.1. *The integral transform*

$$T : u(x) \mapsto \int_{\mathbb{R}^m} K^+(x, x')\frac{u(x')}{|x'|}dx',$$
$$K^+(x, x') := \frac{2^{\frac{m-1}{2}}\psi(x, x')^{-\frac{m-3}{2}}}{e^{\frac{m-1}{2}\pi\sqrt{-1}}\pi^{\frac{m-1}{2}}} J_{\frac{m-3}{2}}(\psi(x, x'))$$

is a unitary operator on $L^2(\mathbb{R}^m, \frac{dx}{|x|})$ of order two (m: odd) and of order four (m: even), that is, we have:

(Inversion formula) $\quad T^{-1} = (-1)^{m+1}T,$

(Plancherel formula) $\quad \|Tu\|_{L^2(\mathbb{R}^m, \frac{dx}{|x|})} = \|u\|_{L^2(\mathbb{R}^m, \frac{dx}{|x|})}$ *for all* $u \in L^2(\mathbb{R}^m, \dfrac{dx}{|x|})$.

Remark 6.2.2 (Weil representation, see [16, Corollaries 5.7.3 and 5.7.4])). *Let us compare Corollary 6.2.1 with the corresponding result for the Schrödinger model $L^2(\mathbb{R}^m)$ of the Weil representation ϖ. The unitary operator $\varpi(w_0)$ corresponding to the "inversion element" w_0 is given by the (ordinary) Fourier transform $\mathcal{F}$ (see Fact D in Subsection 1.2). As is well-known, $\mathcal{F}$ is a unitary operator of order four. This reflects the fact that $w_0^4 = e$ in $Sp(m, \mathbb{R})\tilde{}$.*

6.3 The Hankel transform

Similar to Corollary 6.2.1, from the equality $\pi_+(w_0)^2 = (-1)^{m+1}\mathrm{id}$ and Theorem 6.1.2, we have:

Corollary 6.3.1. *Let ν be a positive integer, and $x, y \in \mathbb{R}$. Then the integral transform*

$$\mathcal{T}_\nu : f(x) \mapsto \int_0^\infty J_\nu(xy) f(y) \sqrt{xy}\, dy$$

is a unitary operator on $L^2((0,\infty), dx)$ of order two. Hence we have:

$$\begin{aligned} &\text{(Inversion formula)} \quad \mathcal{T}_\nu^{-1} = \mathcal{T}_\nu, \\ &\text{(Plancherel formula)} \quad \|\mathcal{T}_\nu f\|_{L^2(\mathbb{R}_+, dx)} = \|f\|_{L^2(\mathbb{R}_+, dx)}. \end{aligned}$$

Remark 6.3.2. *The unitary operator $\mathcal{T}_\nu$ coincides with the Hankel transform (see [7, Chapter VIII]) and the property $\mathcal{T}_\nu^2 = \mathrm{id}$ in the corollary corresponds to its classically known reciprocal formula due to Hankel [13] (see also [7, §8.1 (1)], [33, §14.3 (3)]). The Parseval-Plancherel formula for the Hankel transform goes back to Macaulay-Owen [25].*

Proof of Corollary 6.3.1. Since we assume ν is a positive integer, $\nu = m - 2 + 2l$ for some $m > 3$ and $l \in \mathbb{Z}$. We change the variables $r = \frac{x^2}{4}$, $r' = \frac{y^2}{4}$ and define a unitary map $\Phi : L^2(\mathbb{R}_+, r^{m-2}dr) \to L^2(\mathbb{R}), (\Phi f)(x) := \left(\frac{x}{2}\right)^{\frac{2m-3}{2}} f\left(\frac{x^2}{4}\right)$. Then, we have $\mathcal{T}_\nu = e^{-(l+\frac{m-1}{2})\pi\sqrt{-1}} \Phi \circ T_l \circ \Phi^{-1}$. Since $T_l = \pi_{+,l}(w_0)$ by Theorem 6.1.2, T_l has its inverse as $T_l^{-1} = (-1)^{m+1} T_l$, $\mathcal{T}_\nu$ has its inverse given by

$$\mathcal{T}_\nu^{-1} = e^{(l+\frac{m-1}{2})\pi\sqrt{-1}} \Phi \circ (-1)^{m+1} T_l \circ \Phi^{-1} = \mathcal{T}_\nu.$$

Hence, the corollary follows. $\square$

6.4 Forward and backward light cones

So far, we have discussed only the irreducible unitary representation π_+ realized on $L^2(C_+)$ for the forward light cone. In this subsection, let us briefly comment on $(\pi_-, L^2(C_-))$ for the backward light cone and $(\pi, L^2(C))$ for $C = C_+ \cup C_-$.

Since $d\pi_-(Z) = -d\pi_+(Z)$ in the polar coordinate representation, we can define a semigroup of contraction operators $\{\pi_-(e^{tZ}) : \mathrm{Re}\, t < 0\}$ similarly on $L^2(C_-)$. All the statements about the semigroup $\pi_+(e^{tZ})$ also hold by changing the signature $t \to -t$ and replacing C_+ by C_-.

We define a function $K(\zeta, \zeta')$ on $C \times C$ by

$$K(\zeta, \zeta') := \begin{cases} K^+(\zeta, \zeta') & \text{if } \langle \zeta, \zeta' \rangle \geq 0 \\ 0 & \text{if } \langle \zeta, \zeta' \rangle < 0. \end{cases} \tag{6.4.1}$$

Here we note:

$$\{(\zeta, \zeta') \in C \times C : \langle \zeta, \zeta' \rangle \geq 0\} = (C_+ \times C_+) \cup (C_- \times C_-).$$

Originally, $K^+(\zeta, \zeta')$ was defined on $C_+ \times C_+$ (see (6.1.1)). Since $K^+(\zeta, \zeta')$ depends only on the inner product $\langle \zeta, \zeta' \rangle$, we can define $K^+(\zeta, \zeta')$ also on $C_- \times C_-$.

Corollary 6.4.1. *The unitary operator $\pi(w_0) : L^2(C) \to L^2(C)$ coincides with the integral transform defined by*

$$T : L^2(C) \to L^2(C), \quad u \mapsto \int_C K(\zeta, \zeta') u(\zeta') d\mu(\zeta'). \tag{6.4.2}$$

Remark 6.4.2. *We note that the kernel function $K(\zeta, \zeta')$ is supported on the proper subset $\{(\zeta, \zeta') \in C \times C : \langle \zeta, \zeta' \rangle \geq 0\}$ of $C \times C$. More generally, in the case of the minimal representation of $O(p, q)$ with $p + q$: even, ≥ 8, we shall see in [20] that the integral kernel $K_{p,q}(\zeta, \zeta')$ representing the inversion element is also supported on the proper subset $\{(\zeta, \zeta') \in C \times C : \langle \zeta, \zeta' \rangle \geq 0\}$ if both integers p, q are even. This feature fails if both p, q are odd. In fact, the support of $K_{p,q}$ is the whole space $C \times C$ then.*

7 Explicit actions of the whole group on $L^2(C)$

Building on the explicit formula of $\pi(w_0) = \pi(e^{\pi\sqrt{-1}Z})$ on $L^2(C)$ (see Corollary 6.4.1) and that of $\pi(g)$ ($g \in \overline{P^{\max}}$) (see Subsection 2.2), we can find an explicit formula of the minimal representation for the whole group G.

For simplicity, this section treats the case where m is odd. The main result of this section is Theorem 7.2.1 for the action of $O(m + 1, 2)$.

7.1 Bruhat decomposition of $O(m + 1, 2)$

We recall the notation in Subsection 2.1. In particular, $\overline{P^{\max}} = M^{\max} A \overline{N^{\max}}$ is a maximal parabolic subgroup of $O(m + 1, 2)$. Hence, $O(m + 1, 2)$ is expressed as the disjoint union:

$$\begin{aligned} O(m + 1, 2) &= \overline{P^{\max}} \amalg \overline{P^{\max}} w_0 \overline{P^{\max}} \\ &= \overline{P^{\max}} \amalg \overline{N^{\max}} A M^{\max} w_0 \overline{N^{\max}}. \end{aligned}$$

We begin by finding $a, b \in \mathbb{R}^{m+1}$, $t \in \mathbb{R}$ and $m \in M^{\max}$ such that

$$g = \overline{n}_b e^{tE} m w_0 \overline{n}_a \in \overline{N^{\max}} A M^{\max} w_0 \overline{N^{\max}} \tag{7.1.1}$$

holds for $g \notin \overline{P^{\max}}$.

Suppose g is of the form (7.1.1), and we write $m = \delta m_+$ ($\delta = \pm 1, m_+ \in M_+^{\max}$). Then, in light of the formulas (2.1.3)–(2.1.5), we have

$$\begin{aligned} g(e_0 - e_{m+2}) &= \overline{n}_b e^{tE} (\delta m_+) w_0 \overline{n}_a (e_0 - e_{m+2}) \\ &= \overline{n}_b e^{tE} (\delta m_+) w_0 (e_0 - e_{m+2}) \\ &= \overline{n}_b e^{tE} (\delta m_+) w_0 (e_0 - e_{m+2}) \\ &= \delta e^t \overline{n}_b (e_0 + e_{m+2}) \\ &= \delta e^t \, {}^t(1 - Q(b), 2b_1, \ldots, 2b_{m+1}, 1 + Q(b)). \end{aligned} \tag{7.1.2}$$

Now, we set

$$x \equiv {}^t(x_0, \ldots, x_{m+2}) := g(e_0 - e_{m+2}). \tag{7.1.3}$$

Then, solving (7.1.2), we have

$$\begin{aligned} & x_0 + x_{m+2} = 2\delta e^t, \\ & \frac{x_j}{x_0 + x_{m+2}} = b_j \quad (1 \le j \le m+1). \end{aligned}$$

Likewise, if we set

$$y \equiv {}^t(y_0, \ldots, y_{m+2}) := g^{-1}(e_0 - e_{m+2}), \tag{7.1.4}$$

the following relation:

$$\frac{y_j}{y_0 + y_{m+2}} = -a_j \quad (1 \le j \le m+1)$$

must hold, because $g^{-1} = \overline{n}_{-a} e^{tE} (w_0 m^{-1} w_0^{-1}) w_0 \overline{n}_{-b}$.

Let $\langle \ , \ \rangle$ denote the standard positive definite inner product on $\mathbb{R}^{m+3}$. It follows from $g \in O(m+1, 2)$ that

$$\begin{aligned} x_0 + x_{m+2} &= \langle e_0 + e_{m+2}, g(e_0 - e_{m+2}) \rangle \\ &= \langle {}^t g(e_0 + e_{m+2}), e_0 - e_{m+2} \rangle \\ &= \langle I_{m+1,2} g^{-1} I_{m+1,2} (e_0 + e_{m+2}), e_0 - e_{m+2} \rangle \\ &= \langle y, e_0 + e_{m+2} \rangle \\ &= y_0 + y_{m+2}. \end{aligned}$$

If $x_0 + x_{m+2} \neq 0$, we set

$$a := (a_1, \dots, a_{m+1}), \quad a_j := \frac{-y_j}{y_0 + y_{m+2}} \quad (1 \le j \le m+1), \tag{7.1.5 (a)}$$

$$b := (b_1, \dots, b_{m+1}), \quad b_j := \frac{x_j}{x_0 + x_{m+2}} \quad (1 \le j \le m+1), \tag{7.1.5 (b)}$$

$$\delta := \operatorname{sgn}(x_0 + x_{m+2}), \tag{7.1.5 (c)}$$

$$t := \log \left| \frac{x_0 + x_{m+2}}{2} \right|, \tag{7.1.5 (d)}$$

$$m_+ := \delta e^{-tE} \overline{n}_{-b} g \overline{n}_{-a} w_0^{-1}. \tag{7.1.5 (e)}$$

Lemma 7.1.1. *Retain the above notation.*

1) *For $g \in O(m+1, 2)$, the following three conditions are equivalent:*

   i) $g \notin \overline{P^{\max}}$,

   ii) $x_0 + x_{m+2} \neq 0$,

   iii) $y_0 + y_{m+2} \neq 0$.

2) *Suppose one of (therefore, all of) the above conditions holds. Then, the element m_+ defined by* (7.1.5) (e) *belongs to $M_+^{\max} \simeq O(m, 1)$.*

7.2 Explicit action of the whole group

For $g \in O(m+1, 2)$, we set

$$x \equiv {}^t(x_0, \dots, x_{m+2}) := g(e_0 - e_{m+2}) \quad \text{(see (7.1.3))},$$
$$y \equiv {}^t(y_0, \dots, y_{m+2}) := g^{-1}(e_0 - e_{m+2}) \quad \text{(see (7.1.4))}.$$

For g such that $x_0 + x_{m+2} \neq 0$, we also set $a = (a_1, \dots, a_{m+1})$, $b = (b_1, \dots, b_{m+1})$, $\delta \in \{\pm 1\}$, and m_+ as in (7.1.5).

If $x_0 + x_{m+2} = 0$, then $g \in \overline{P^{\max}}$ and $(\pi(g)\psi)(\zeta)$ is obtained readily by the formulas (2.2.2)–(2.2.5). For generic g such that $x_0 + x_{m+2} \neq 0$, the unitary operator $\pi(g)$ can be written by means of the kernel function $K(\zeta, \zeta')$ (see (6.4.1) for the definition), the above $x, y \in \mathbb{R}^{m+3}$ and $m_+ \in M_+^{\max} \simeq O(m, 1)$ as follows:

Theorem 7.2.1. *For $g \in O(m+1, 2)$ such that $x_0 + x_{m+2} \neq 0$, the unitary operator $\pi(g)$ is given by the following integral formula: for $\psi \in L^2(C)$,*

$$(\pi(g)\psi)(\zeta) = \left(\frac{2}{x_0 + x_{m+2}} \right)^{\frac{m-1}{2}} e^{\frac{2\sqrt{-1}(x_1\zeta_1 + \cdots + x_{m+1}\zeta_{m+1})}{x_0 + x_{m+2}}}$$
$$\int_C K\left(\frac{2\zeta}{|x_0 + x_{m+2}|}, m_+\zeta' \right) e^{\frac{-2\sqrt{-1}(y_1\zeta_1' + \cdots + y_{m+1}\zeta_{m+1}')}{y_0 + y_{m+2}}} \psi(\zeta') du(\zeta').$$

Proof. We write $g = \overline{n}_b e^t(\delta m_+)w_0\overline{n}_a$ as in the form (7.1.1). Then, we have

$$
\begin{aligned}
&(\pi(g)\psi)(\zeta) \\
&= \pi(\overline{n}_b)\pi(e^{tE})\pi(\delta m_+)(\pi(w_0\overline{n}_a)\psi)(\zeta) \\
&= e^{2\sqrt{-1}\langle b,\zeta\rangle} e^{-\frac{m-1}{2}t}\delta^{\frac{m-1}{2}}(\pi(w_0\overline{n}_a)\psi)(e^{-t}\,{}^t m_+\zeta) \quad \text{by (2.2.2)–(2.2.5).}
\end{aligned}
$$

Now, by using Corollary 6.4.1,

$$
\begin{aligned}
&= e^{2\sqrt{-1}\langle b,\zeta\rangle} e^{-\frac{m-1}{2}t}\delta^{\frac{m-1}{2}} \int_C K(e^{-t}\,{}^t m_+\zeta, \zeta')(\pi(\overline{n}_a)\psi)(\zeta')du(\zeta') \\
&= e^{2\sqrt{-1}\langle b,\zeta\rangle} e^{-\frac{m-1}{2}t}\delta^{\frac{m-1}{2}} \int_C K(e^{-t}\zeta, m_+\zeta')e^{2\sqrt{-1}\langle a,\zeta'\rangle}\psi(\zeta')du(\zeta').
\end{aligned}
$$

□

8 Appendix: special functions

For the convenience of the reader, we collect here basic formulas of special functions in a way that we use in this article.

8.1 Laguerre polynomials

For $\alpha \in \mathbb{C}, n \in \mathbb{N}$, the *Laguerre polynomials* $L_n^\alpha(x)$ are defined by the formula (see [1, §6.2], for example):

$$
\begin{aligned}
L_n^\alpha(x) &:= \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}) \qquad (8.1.1)\\
&= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k x^k}{(\alpha+1)_k k!} \\
&= \frac{(-1)^n}{n!}x^n + \cdots + \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.
\end{aligned}
$$

Here, we write β_n for $\beta(\beta+1)\cdots(\beta+n-1)$. The Laguerre polynomial solves the linear ordinary differential equation of second order:

$$xu'' + (\alpha+1-x)u' + nu = 0. \qquad (8.1.2)$$

Suppose $\alpha \in \mathbb{R}$ and $\alpha > -1$. Then the Laguerre polynomials $\{L_n^\alpha(x) : n = 0, 1, \cdots\}$ are complete in $L^2((0,\infty), x^\alpha e^{-x}dx)$, and satisfy the orthogonality relation (see [1, §6.5]):

$$\int_0^\infty L_m^\alpha(x)L_n^\alpha(x)x^\alpha e^{-x}dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{mn} \quad (\alpha > -1). \qquad (8.1.3)$$

The Hille–Hardy formula gives the bilinear generating function of Laguerre polynomials (see [6, §10.12 (20)], [14, §1 (3)]):

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x) L_n^{\alpha}(y) w^n$$
$$= \frac{1}{1-w} \exp\Big(-\frac{(x+y)w}{1-w}\Big)(-xyw)^{-\frac{\alpha}{2}} J_{\alpha}\Big(\frac{2\sqrt{-xyw}}{1-w}\Big), \quad x>0, y>0, |w|<1. \tag{8.1.4}$$

Here the left-hand side converges absolutely.

8.2 Hermite polynomials

Hermite polynomials $H_n(x)$ are given as special values of Laguerre polynomials. We recall from [6, II, §10.13] that

$$H_{2m}(x) = (-1)^m 2^{2m} m!\, L_m^{-\frac{1}{2}}(x^2), \tag{8.2.1}$$
$$H_{2m+1}(x) = (-1)^m 2^{2m+1} m!\, x L_m^{\frac{1}{2}}(x^2), \tag{8.2.2}$$

This reduction formula is reflected by the fact that Hermite polynomials appear in the analysis of the Weil representation of $SL(2,\mathbb{R})^{\sim}$, while Laguerre polynomials appear in the analysis of the minimal representation of $SO_0(m+1,2)^{\sim}$ and $Sp(m,\mathbb{R})^{\sim}$ (see Remark 4.3.2).

8.3 Gegenbauer polynomials

For $\nu \in \mathbb{C}$ and $l \in \mathbb{N}$, the Gegenbauer polynomials $C_l^{\nu}(x)$ are the polynomials of x of degree l given by the *Rodrigues formula* (see [1, §6.4]):

$$C_l^{\nu}(x) := \big(-\frac{1}{2}\big)^l \frac{(2\nu)_l}{l!\,(\nu+\frac{1}{2})_l} (1-x^2)^{-\nu+\frac{1}{2}} \frac{d^l}{dx^l}\big((1-x^2)^{l+\nu-\frac{1}{2}}\big). \tag{8.3.1}$$

It then follows from (8.3.1) that

$$C_l^{\nu}(1) = \frac{\Gamma(2\nu+l)}{l!\,\Gamma(2\nu)}.$$

We renormalize the Gegenbauer polynomial by

$$\widetilde{C}_l^{\nu}(x) := \Gamma(\nu) C_l^{\nu}(x). \tag{8.3.2}$$

Then by the duplication formula of the Gamma function:

$$\Gamma(2\nu) = \frac{2^{2\nu-1}}{\sqrt{\pi}}\Gamma(\nu)\Gamma(n+\frac{1}{2}),$$

we have

$$\widetilde{C}_l^\nu(1) = \frac{\sqrt{\pi}\,\Gamma(2\nu+l)}{2^{2\nu-1}l!\,\Gamma(\nu+\frac{1}{2})}. \tag{8.3.3}$$

The special value at $\nu = 0$ is given by the limit formula (see [6, **I**, §3.15.1 (14)])

$$\widetilde{C}_l^0(\cos\theta) = \lim_{\nu\to 0}\Gamma(\nu)C_l^\nu(\cos\theta) = \frac{2\cos(l\theta)}{l}.$$

Suppose $\operatorname{Re}\nu > -\frac{1}{2}$. Then, $\{\widetilde{C}_l^\nu(x) : l = 0, 1, 2, \ldots\}$ forms a complete orthogonal basis of $L^2((-1,1),(1-x^2)^{\nu-\frac{1}{2}}dx)$, and the norm of $\widetilde{C}_l^\nu(x)$ is given by

$$\int_{-1}^{1}\widetilde{C}_l^\nu(x)^2(1-x^2)^{\nu-\frac{1}{2}}dx = \frac{2^{1-2\nu}\pi\Gamma(2\nu+l)}{l!\,(l+\nu)}, \tag{8.3.4}$$

(see [6, **I**, §3.15.1 (17)]). Therefore, $f \in L^2((-1,1),(1-x^2)^{\nu-\frac{1}{2}}dx)$ has the following expansion:

$$f(x) = \sum_{l=0}^{\infty}\alpha_l^\nu(f)\widetilde{C}_l^\nu(x), \tag{8.3.5}$$

where we set

$$\alpha_l^\nu(f) := \frac{l!\,(l+\nu)}{2^{1-2\nu}\pi\Gamma(2\nu+l)}\int_{-1}^{1} f(x)\widetilde{C}_l^\nu(x)(1-x^2)^{\nu-\frac{1}{2}}dx. \tag{8.3.6}$$

The following integral formula is used in the proof of Lemma 8.5.2 (see [7, §16.3 (3)]): Suppose $\operatorname{Re}\beta > -1$ and $\operatorname{Re}\nu > -\frac{1}{2}$.

$$\int_{-1}^{1}(1-x)^{\nu-\frac{1}{2}}(1+x)^\beta\widetilde{C}_n^\nu(x)dx = \frac{2^{\beta-\nu+\frac{3}{2}}\sqrt{\pi}\,\Gamma(\beta+1)\Gamma(2\nu+n)\Gamma(\beta-\nu+\frac{3}{2})}{n!\,\Gamma(\beta-\nu-n+\frac{3}{2})\Gamma(\beta+\nu+n+\frac{3}{2})}. \tag{8.3.7}$$

8.4 Spherical harmonics and Gegenbauer polynomials

Let $\Delta_{S^{n-1}}$ be the Laplace–Beltrami operator on the $(n-1)$-dimensional unit sphere S^{n-1}. Then the *spherical harmonics* on S^{n-1} are defined as

$$\mathcal{H}^l(\mathbb{R}^n) := \{f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}}f = -l(l+n-2)f\}, \quad l = 0, 1, 2, \ldots.$$

The following facts are well-known:

(1) $\mathcal{H}^l(\mathbb{R}^n)$ is an irreducible representation space of $O(n)$.

(2) It is still irreducible as an $SO(n)$-module if $n \geq 3$.

(3) $\mathcal{H}^l(\mathbb{R}^2) = \mathbb{C}e^{\sqrt{-1}l\theta} \oplus \mathbb{C}e^{-\sqrt{-1}l\theta}$, $l \geq 1$ as $SO(2)$-modules, where $\theta = \tan^{-1}\frac{x_2}{x_1}$, $(x_1, x_2) \in \mathbb{R}^2$.

(4) $\mathcal{H}^l(\mathbb{R}^n)\big|_{O(n-1)} \simeq \bigoplus_{k=0}^{l} \mathcal{H}^k(\mathbb{R}^{n-1})$ gives an irreducible decomposition of $O(n-1)$-modules. This is also an irreducible decomposition as $SO(n-1)$-modules ($n \geq 4$).

We regard $O(n-1)$ as the isotropy subgroup of $O(n)$ at $(1, 0, \cdots, 0) \in \mathbb{R}^n$. We write $(x_1, \cdots, x_n)$ for the standard coordinate of $\mathbb{R}^n$. Then, $O(n-1)$-invariant spherical harmonics are unique up to scalar, and we have:

$$\mathcal{H}^l(\mathbb{R}^n)^{O(n-1)} \simeq \mathbb{C}\widetilde{C}_l^{\frac{n-2}{2}}(x_1). \tag{8.4.1}$$

8.5 Bessel functions

For $\nu \in \mathbb{C}, z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, *Bessel functions* $J_\nu(z)$ are defined by

$$J_\nu(z) := \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m}}{m!\,\Gamma(\nu + m + 1)}. \tag{8.5.1}$$

It solves the *Bessel's differential equation* of second order:

$$u'' + \frac{1}{z}u' + \left(1 - \frac{\nu^2}{z^2}\right)u = 0.$$

The modified Bessel functions $I_\nu(z)$ are defined by

$$I_\nu(z) := \begin{cases} e^{-\frac{\pi\nu\sqrt{-1}}{2}} J_\nu(\sqrt{-1}z) & (-\pi < \arg z \leq \frac{\pi}{2}), \\ e^{\frac{3\pi\nu\sqrt{-1}}{2}} J_\nu(\sqrt{-1}z) & (\frac{\pi}{2} < \arg z < \pi). \end{cases} \tag{8.5.2}$$

$$= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\,\Gamma(\nu + m + 1)} \quad (z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}). \tag{8.5.3}$$

For a special value $\nu = \pm\frac{1}{2}$, $I_\nu(z)$ reduces to

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh z, \quad I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cosh z. \tag{8.5.4}$$

We set

$$\tilde{J}_\nu(z) := \left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{m!\,\Gamma(\nu+m+1)}, \tag{8.5.5}$$

$$\tilde{I}_\nu(z) := \left(\frac{z}{2}\right)^{-\nu} I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\,\Gamma(\nu+m+1)}. \tag{8.5.6}$$

We note that $\tilde{J}_\nu(z)$ and $\tilde{I}_\nu(z)$ are entire functions of z, and

$$\tilde{J}_\nu(\sqrt{-1}z) = \tilde{I}_\nu(z).$$

The following lemma on the estimate of I-Bessel functions are used in Subsections 4.4 and 5.2. We need an estimate of $\tilde{I}_\nu(z)$ for $\nu \geq -\frac{1}{2}$:

Lemma 8.5.1. *There exists a constant $C > 0$ such that the following estimate holds for all $z \in \mathbb{C}$:*

$$|\tilde{I}_\nu(z)| \leq Ce^{|\operatorname{Re} z|}. \tag{8.5.7}$$

Proof. First, suppose $\nu = -\frac{1}{2}$. Then $|\tilde{I}_\nu(z)| = \frac{1}{\sqrt{\pi}}|\cosh z| \leq \frac{1}{\sqrt{\pi}}e^{|\operatorname{Re} z|}$.

Next, suppose $\nu > -\frac{1}{2}$. By an integral representation of the Bessel function [33, §6.15 (2)]:

$$\tilde{I}_\nu(z) = \frac{1}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{-zt}(1-t^2)^{\nu-\frac{1}{2}}dt,$$

we have

$$\begin{aligned} |\tilde{I}_\nu(z)| \leq& \frac{1}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{-t \operatorname{Re} z}(1-t^2)^{\operatorname{Re}\nu-\frac{1}{2}}dt \\ \leq& \frac{e^{|\operatorname{Re} z|}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\operatorname{Re}\nu-\frac{1}{2}}dt \leq Ce^{|\operatorname{Re} z|} \end{aligned}$$

for some constant C independent of z. □

The following lemma is used in Subsection 5.5, where we set $\nu = \frac{m-3}{2}, \alpha = \frac{2\sqrt{2rr'}}{\sinh\frac{t}{2}}$.

Lemma 8.5.2. *Assume $\alpha \in \mathbb{C}, \nu \geq -\frac{1}{2}$, and $l \in \mathbb{N}$. Then we have:*

$$\begin{aligned} &\int_0^{\pi} I_\nu\big(\alpha\sqrt{1+\cos\theta}\big)\widetilde{C}_l^{\nu+\frac{1}{2}}(\cos\theta)(1+\cos\theta)^{-\frac{\nu}{2}}\sin^{2\nu+1}\theta d\theta \\ &\qquad = \frac{2^{\frac{3}{2}}\sqrt{\pi}\,\Gamma(2\nu+l+1)}{\alpha^{\nu+1}l!} I_{2\nu+2l+1}\big(\sqrt{2}\alpha\big). \end{aligned} \tag{8.5.8}$$

We could not find this formula in the literature, and so we give its proof here.

Proof. First we note that the integral (8.5.8) converges since $I_\nu\big(\alpha(\sqrt{1+\cos\theta})\big)\cdot(1+\cos\theta)^{-\frac{\nu}{2}}$ is continuous on the closed interval $[0,\pi]$. Furthermore, we have a uniformly convergent expansion

$$I_\nu\big(\alpha(\sqrt{1+\cos\theta})\big)\cdot(1+\cos\theta)^{-\frac{\nu}{2}}=\sum_{j=0}^{\infty}\frac{\left(\frac{\alpha}{2}\right)^{\nu+2j}(1+\cos\theta)^j}{j!\,\Gamma(j+\nu+1)}.$$

Now the left-hand side of (8.5.8) equals

$$\begin{aligned}
&\left(\frac{\alpha}{2}\right)^\nu\sum_{j=0}^{\infty}\frac{\left(\frac{\alpha}{2}\right)^{2j}}{j!\,\Gamma(j+\nu+1)}\int_0^\pi(1+\cos\theta)^j\widetilde{C}_l^{\nu+\frac{1}{2}}(\cos\theta)\sin^{2\nu+1}\theta d\theta\\
&=\left(\frac{\alpha}{2}\right)^\nu\sum_{j=0}^{\infty}\frac{\left(\frac{\alpha}{2}\right)^{2j}}{j!\,\Gamma(j+\nu+1)}\frac{2^{j+1}\sqrt{\pi}\,\Gamma(j+\nu+1)\Gamma(2\nu+l+1)\Gamma(j+1)}{l!\,\Gamma(j-l+1)\Gamma(2\nu+j+l+2)}\\
&=\left(\frac{\alpha}{2}\right)^\nu\frac{2\sqrt{\pi}\,\Gamma(2\nu+l+1)}{l!}\sum_{j=0}^{\infty}\frac{\left(\frac{\alpha}{\sqrt{2}}\right)^{2j}}{\Gamma(j-l+1)\Gamma(2\nu+j+l+2)}\\
&=\left(\frac{\alpha}{2}\right)^\nu\frac{2\sqrt{\pi}\,\Gamma(2\nu+l+1)}{l!}\left(\frac{\alpha}{\sqrt{2}}\right)^{2l}\sum_{n=0}^{\infty}\frac{\left(\frac{\alpha}{\sqrt{2}}\right)^{n}}{\Gamma(n+1)\Gamma(2\nu+2l+n+2)}\\
&=\text{the right-hand side of (8.5.8).}
\end{aligned}$$

Here the first equality follows from the formula (8.3.7). In fact, the substitution of $x=\cos\theta$ into (8.3.7) yields

$$\begin{aligned}
&\int_0^\pi(1+\cos\theta)^j\widetilde{C}_l^{\nu+\frac{1}{2}}(\cos\theta)\sin^{2\nu+1}\theta d\theta\\
&=\int_{-1}^{1}(1-x)^\nu(1+x)^{j+\nu}\widetilde{C}_l^{\nu+\frac{1}{2}}(x)dx\\
&=\frac{2^{j+1}\sqrt{\pi}\,\Gamma(j+\nu+1)\Gamma(2\nu+l+1)\Gamma(j+1)}{l!\,\Gamma(j-l+1)\Gamma(2\nu+j+l+2)}.
\end{aligned}$$

Thus, the lemma has been proved. □

References

[1] G. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge, 1999.

[2] W. Beckner, Inequalities in Fourier analysis, *Ann. of Math.*, (2) **102** (1975), 159–182.

[3] B. Binegar and R. Zierau, Unitarization of a singular representation of $SO(p,q)$, *Comm. Math. Phys.*, **138** (1991), 245–258.

[4] H. Ding, K. I. Gross, R. A. Kunze, and D. St. P. Richards, Bessel functions on boundary orbits and singular holomorphic representations, *The mathematical legacy of Harish-Chandra* (Baltimore, MD, 1998), 223–254, Proc. Sympos. Pure Math., **68**, Amer. Math. Soc., Providence, RI, 2000.

[5] A. Dvorsky and S. Sahi, Explicit Hilbert spaces for certain unipotent representations II, *Invent. Math.*, **138** (1999), 203–224.

[6] A. Erdélyi et al., *Higher Transcendental Functions* **I**, **II**, McGraw-Hill, New York, 1953.

[7] A. Erdélyi et al., *Tables of Integral Transforms*, **II**, McGraw-Hill, New York, 1954.

[8] B. Folland, *Harmonic Analysis in Phase Space*, Annals of Mathematics Studies, **122**, Princeton University Press, Princeton, NJ, 1989.

[9] L. Gegenbauer, Über die Functionen X_n^m, *Wiener Sitzungsberichte*, **68** (2) (1874), 357–367.

[10] I. M. Gelfand and S. G. Gindikin, Complex manifolds whose skeltons are real semisimple groups, and the holomorphic discrete series, *Funct. Anal. Appl.*, **11** (1977), 19–27.

[11] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, **I**, Academic Press, New York, 1964.

[12] V. Guillemin and S. Sternberg, Variations on a Theme by Kepler, *Amer. Math. Soc. Colloq. Publ.*, **42**, Amer. Math. Soc., Province, 1990.

[13] H. Hankel, Die Fourier'schen Reihen und Integrale für Cylinderfunctionen, *Math. Ann.*, **8** (1875), 471–494.

[14] G. H. Hardy, Summation of a series of polynomials of Laguerre, Journal of the London Mathematical Society, **7** (1932), 138–139, 192.

[15] S. Helgason, *Groups and Geometric Analysis*, Academic Press, London, 1984.

[16] R. Howe, The oscillator semigroup, Amer. Math. Soc., Proc. Symp. Pure Math., **48** (1988), 61–132.

[17] R. Howe and E. C. Tan, *Non-Abelian Harmonic Analysis*, Springer, 1992.

[18] T. Kobayashi, Conformal geometry and global solutions to the Yamabe equations on classical pseudo-Riemannian manifolds, Proceedings of the 22nd Winter School "Geometry and Physics" (Srni, 2002). Rend. Circ. Mat. Palermo (2) Suppl., **71** (2003), 15–40.

[19] T. Kobayashi and G. Mano, Integral formulas for the minimal representation of $O(p, 2)$, *Acta Appl. Math.*, **86** (2005), 103–113.

[20] T. Kobayashi and G. Mano, Inversion formula for the minimal representation of $O(p, q)$, in preparation.

[21] T. Kobayashi and B. Ørsted, Analysis on the minimal representation of $O(p, q)$ I. Realization via conformal geometry, *Adv. Math.*, **180** (2003), 486–512.

[22] T. Kobayashi and B. Ørsted, Analysis on the minimal representation of $O(p, q)$ II. Branching laws, *Adv. Math.*, **180** (2003), 513–550.

[23] T. Kobayashi and B. Ørsted, Analysis on the minimal representation of $O(p, q)$ III. Ultrahyperbolic equations on $\mathbb{R}^{p-1,q-1}$, *Adv. Math.*, **180** (2003), 551–595.

[24] B. Kostant, The vanishing scalar curvature and the minimal unitary representation of $SO(4, 4)$, eds. Connes et al, *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory*, Progress in Math., **92** , Birkhäuser, 1990, 85–124.

[25] P. Macaulay-Owen, Parseval's theorem for Hankel transforms, *Proc. London Math. Soc.*, (2) **45** (1939), 458–474.

[26] W. Myller-Lebedeff, Die Theorie der Integralgleichungen in Anwendung auf einige Reihenentwicklungen, *Math. Ann.*, **68** (1907), 388–416.

[27] G. I. Olshanskiĭ, Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series, *Funct. Anal. Appl.*, **15** (1981), 275–285.

[28] S. Sahi, Explicit Hilbert spaces for certain unipotent representations, *Invent. Math.*, **110** (1992), 409–418.

[29] R. J. Stanton, Analytic extension of the holomorphic discrete series, *Amer. J. Math.*, **10** (1986), 1411–1424.

[30] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, Province, 1939.

[31] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, Mathematical Notes **42**, Princeton University Press, Princeton, 1993.

[32] P. Torasso, Méthode des orbites de Kirillov–Duflo et représentations minimales des groupes simples sur un corps local de caractéristique nulle, *Duke Math. J.*, **90** (1997), 261–377.

[33] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge, 1922.